SPHERICAL ASTRONOMY.

AN

ELEMENTARY TREATISE

ON

SPHERICAL ASTRONOMY:

ADAPTED TO A

COURSE OF INSTRUCTION

IN

CIVIL ENGINEERING.

BY

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PREFACE.

The following course in Spherical Astronomy has been prepared for the use of the Author's classes, and is intended to include those applications of astronomy which fall within the province of the Civil Engineer. It assumes a preliminary knowledge of general Descriptive Astronomy, and is also designed to be supplemented by a course of instruction and practice in the adjustment and use of portable astronomical instruments, and in practical computations. It is for this reason that theoretical solutions only have been given of the various problems considered, all examples having been omitted; but the development of the working formulæ has been carried to the point required for their practical application, and the results are given in a form adapted to immediate use.

CONTENTS.

SPHERICAL ASTRONOMY.

CHAPTER I.

SPHERICAL PROBLEMS.

1. In *Spherical Astronomy* the real *distances* and *magnitudes* of the celestial bodies are not considered, but only their relative *directions*. Hence, whatever may be their actual distances from the observer, they are all regarded as situated on the surface of a *Celestial Sphere* of indefinitely great radius, of which the earth is the center.

2. The fundamental definitions of Astronomy are illustrated in Fig. 1, which represents the principal circles of the celestial sphere projected on the plane of the meridian.

The observer being supposed to be in north latitude, HZR is the *meridian*, HAR the *horizon*, ZA the *prime vertical*, EQ the *equator*, CD the *ecliptic*, V the *vernal equinox*, Z the *zenith*, P the *north pole*, H the *north point*, R the *south point*, S the *place of a star*, ZO the star's *vertical circle*, PM its *hour circle*, and SL its *circle of latitude*.

3. The co-ordinates which determine the position of a celestial body and that of the observer, are reoresented by the following notation:

$ZE = \phi =$ latitude of the place,
$PZ = \psi =$ colatitude of do.,
$SO = h =$ star's altitude,
$ZS = z =$ " zenith distance,
$PZS = HO = Z =$ " azimuth from north point,
$SZR = OR = Z' =$ " " from south point,
$ZPS = EM = P =$ " hour angle,
$ZSP = S =$ " parallactic angle,
$AO = a =$ " amplitude,
$VM = \alpha =$ " right ascension,
$MS = \delta =$ " declination,
$PS = p =$ " polar distance,
$VL = L =$ " longitude,
$LS = \lambda =$ " latitude,
$CVE = \omega =$ obliquity of the ecliptic.

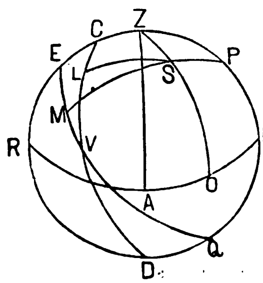

FIG. 1.

4. Since PE, ZO, PM, HA and RA are quadrants, we have

$$\psi = 90° - \phi \quad (1)$$
$$z = 90° - h \quad (2)$$
$$p = 90° - \delta \quad (3)$$
$$a = 90° - Z \quad (4)$$
$$a = Z' - 90° \quad (5)$$
whence $Z' = 180° - Z \quad (6)$

5. Many of the most important problems of Spherical Astronomy can be reduced to the solution of the spherical triangle PZS, Fig. 2, formed by joining the pole, the zenith and the place of a star, by arcs of great circles.

The three sides of this triangle are

$$PZ = 90^\circ - \phi = \text{colatitude},$$
$$PS = 90^\circ - \delta = \text{star's polar distance},$$
$$ZS = 90^\circ - h = \text{" zenith distance},$$

and the three angles are

$P =$ star's hour angle,
$Z =$ " azimuth from north point,
$S =$ " parallactic angle.

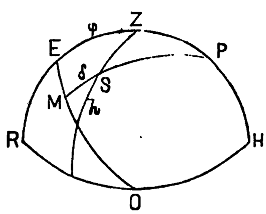

FIG. 2.

6. The following well-known formulæ of Spherical Trigonometry, applied to the triangle PZS, will furnish most of the general equations required in the discussions which follow. Denoting the angles of any spherical triangle by A, B, C, and its sides by a, b, c, we have

$$\left.\begin{aligned} \sin a \sin B &= \sin b \sin A \\ \sin b \sin C &= \sin c \sin B \\ \sin c \sin A &= \sin a \sin C \end{aligned}\right\} \quad (7)$$

$$\left.\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos A \\ \cos b &= \cos c \cos a + \sin c \sin a \cos B \\ \cos c &= \cos a \cos b + \sin a \sin b \cos C \end{aligned}\right\} \quad (8)$$

$$\left.\begin{aligned} \sin a \cos B &= \sin c \cos b - \cos c \sin b \cos A \\ \sin b \cos C &= \sin a \cos c - \cos a \sin c \cos B \\ \sin c \cos A &= \sin b \cos a - \cos b \sin a \cos C \end{aligned}\right\} \quad (9)$$

$$\left.\begin{aligned} \sin^2 \tfrac{1}{2} A &= \frac{\sin (s-b) \sin (s-c)}{\sin b \sin c} \\ \sin^2 \tfrac{1}{2} B &= \frac{\sin (s-c) \sin (s-a)}{\sin c \sin a} \end{aligned}\right\} \quad (10)$$

in which $\qquad s = \frac{1}{2}(a + b + c).$

7. If we apply formulæ (7), (8) and (9) to the triangle PZS, making $A = P$, $B = Z$, $C = S$, $a = 90° - h$, $b = 90° - \delta$, $c = 90° - \phi$, we shall obtain the following

General Astronomical Formulæ.

$$\cos h \sin Z = \cos \delta \sin P \tag{11}$$
$$\cos \delta \sin S = \cos \phi \sin Z \tag{12}$$
$$\cos \phi \sin P = \cos h \sin S \tag{13}$$

$$\sin h = \sin \delta \sin \phi + \cos \delta \cos \phi \cos P \tag{14}$$
$$\sin \delta = \sin \phi \sin h + \cos \phi \cos h \cos Z \tag{15}$$
$$\sin \phi = \sin h \sin \delta + \cos h \cos \delta \cos S \tag{16}$$

$$\cos h \cos Z = \sin \delta \cos \phi - \cos \delta \sin \phi \cos P \tag{17}$$
$$\cos \delta \cos S = \sin \phi \cos h - \cos \phi \sin h \cos Z \tag{18}$$
$$\cos \phi \cos P = \sin h \cos \delta - \cos h \sin \delta \cos S \tag{19}$$

By making the proper substitutions in these equations we may find the formulæ for a body in any position in the heavens.

8. Given the *latitude of the place* and the *declination of the body*, to find its *altitude* and *azimuth* when it is on the *six hour circle.*

In this position the hour angle $P = 6$ *hours* $= 90°$, hence $\sin P = 1$, $\cos P = 0$, and (14) becomes

$$\sin h = \sin \delta \sin \phi \tag{20}$$

(11) becomes $\cos h \sin Z = \cos \delta$

(17) becomes $\cos h \cos Z = \sin \delta \cos \phi$

whence by division,

$$\tan Z = \frac{\cot \delta}{\cos \phi} \tag{21}$$

Eqs. (20) and (21) are the expressions required.

9. Given the same data, to find the *hour angle* and *azimuth* of a body in the *horizon.*

In this position, $h = 0°$, $\sin h = 0$, $\cos h = 1$, and by (14),

$$\cos P = -\frac{\sin \delta \sin \phi}{\cos \delta \cos \phi} = -\tan \delta \tan \phi \qquad (22)$$

and by (15),

$$\cos Z = \frac{\sin \delta}{\cos \phi} \qquad (23)$$

10. Given the same data, to find the *altitude* or *zenith distance* of a body on the *meridian*.

In respect to the position of a body on the meridian, there may be three cases:

1*st*. If the body is *south of the zenith*, the point S, Fig. 2, will lie somewhere on the arc ZR, and in the triangle PZS we shall have the angles $S = 0°$, $P = 0°$, whence $\cos S = 1$, $\cos P = 1$, and (16) and (19) become

$$\sin \phi = \sin h \sin \delta + \cos h \cos \delta = \cos (h - \delta)$$
$$\cos \phi = \sin h \cos \delta - \cos h \sin \delta = \sin (h - \delta)$$

from which it appears that $(h - \delta)$ is the complement of (ϕ), that is,

$$h - \delta = 90° - \phi = \psi, \text{ whence } h = \psi + \delta \qquad (24)$$

or $\qquad \phi = 90° - (h - \delta) = z + \delta,$

whence $\qquad z = \phi - \delta \qquad (25)$

2*d*. If the body is *between the zenith and the pole*, the point S, Fig. 2, lies on the arc PZ, and the angles $P = 0°$, $Z = 0°$, whence $\cos P = 1$, $\cos Z = 1$, and (14) and (17) become

$$\sin h = \sin \delta \sin \phi + \cos \delta \cos \phi = \cos (\delta - \phi)$$
$$\cos h = \sin \delta \cos \phi - \cos \delta \sin \phi = \sin (\delta - \phi)$$

hence $(\delta - \phi)$ is the complement of (h), that is

$$\delta - \phi = 90° - h = z \qquad (26)$$

or $\qquad h = 90° - (\delta - \phi) = \phi + p \qquad (27)$

3*d*. If the body is *below the pole*, the point S, Fig. 2, is on the arc PH, and the angles $Z = 0$, $S = 0°$, whence $\cos Z = 1$, $\cos S = 1$, and (15) and (18) become

$$\sin \delta = \sin \phi \sin h + \cos \phi \cos h = \cos (\phi - h)$$
$$\cos \delta = \sin \phi \cos h - \cos \phi \sin h = \sin (\phi - h)$$

hence $(\phi - h)$ is the complement of (δ), that is,

$$\phi - h = 90^\circ - \delta = p, \text{ whence } \quad h = \phi - p \qquad (28)$$

also

$$90^\circ - h = z = 180^\circ - (\phi + \delta) \qquad (29)$$

11. Given the *latitude of the place*, and the *declination* and *zenith distance* of a body, to find its *hour angle* and *azimuth*.

Applying Eqs. (10) to the triangle PZS, Fig. 2, making $a = z$ instead of $90^\circ - h$, as before, we find

$$\sin^2 \tfrac{1}{2} P = \frac{\sin [s - (90^\circ - \delta)] \sin [s - (90^\circ - \phi)]}{\cos \delta \cos \phi} \qquad (30)$$

$$\sin^2 \tfrac{1}{2} Z = \frac{\sin [s - (90^\circ - \phi)] \sin (s - z)}{\cos \phi \sin z} \qquad (31)$$

in which

$$s = \frac{z + (90^\circ - \delta) + (90^\circ - \phi)}{2} = 90^\circ + \tfrac{1}{2} (z - \phi - \delta)$$

Hence

$$s - z = 90^\circ + \tfrac{1}{2} (z - \phi - \delta) - z = 90^\circ - \tfrac{1}{2} (z + \phi + \delta)$$
$$s - (90^\circ - \phi) = \tfrac{1}{2} (z - \phi - \delta) + \phi = \tfrac{1}{2} (z + \phi - \delta)$$
$$s - (90^\circ - \delta) = \tfrac{1}{2} (z - \phi - \delta) + \delta = \tfrac{1}{2} (z - \phi + \delta)$$

By the substitution of these values, (30) and (31) become

$$\sin^2 \tfrac{1}{2} P = \frac{\sin \tfrac{1}{2} (z - \phi + \delta) \sin \tfrac{1}{2} (z + \phi - \delta)}{\cos \delta \cos \phi} \qquad (32)$$

$$\sin^2 \tfrac{1}{2} Z = \frac{\sin \tfrac{1}{2} (z + \phi - \delta) \cos \tfrac{1}{2} (z + \phi + \delta)}{\cos \phi \sin z} \qquad (33)$$

the expressions required.

12. Given the *latitude of the place*, and the *hour angle* and *declination* of a body, to find its *azimuth* and *altitude*.

Suppose the azimuth to be reckoned from the south point. We find from (6), $Z = 180^\circ - Z'$, whence $\sin Z = \sin Z'$, $\cos Z = - \cos Z'$, which values substituted in (11), (14) and (17), reduce them to

$$\cos h \sin Z' = \cos \delta \sin P \tag{34}$$

$$\sin h = \sin \delta \sin \phi + \cos \delta \cos \phi \cos P \tag{35}$$

$$\cos h \cos Z' = -\sin \delta \cos \phi + \cos \delta \sin \phi \cos P \tag{36}$$

Make $$\sin \delta = m \sin M \tag{37}$$

and $$\cos \delta \cos P = m \cos M \tag{38}$$

then (35) and (36) become

$$\begin{aligned} \sin h &= m\,(\sin \phi \sin M + \cos \phi \cos M) \\ &= m \cos (\phi - M) \qquad (39) \\ \cos h \cos Z' &= m\,(\sin \phi \cos M - \cos \phi \sin M) \\ &= m \sin (\phi - M) \qquad (40) \end{aligned}$$

Dividing (37) by (38),

$$\tan M = \frac{\tan \delta}{\cos P} \tag{41}$$

Dividing (34) by (40),

$$\tan Z' = \frac{\cos \delta}{m} \cdot \frac{\sin P}{\sin (\phi - M)},$$

but from (38), $$\frac{\cos \delta}{m} = \frac{\cos M}{\cos P},$$

hence $$\tan Z' = \frac{\cos M \tan P}{\sin (\phi - M)} \tag{42}$$

Dividing (40) by (39),

$$\frac{\cos Z'}{\tan h} = \tan (\phi - M)$$

whence $$\tan h = \frac{\cos Z'}{\tan (\phi - M)} \tag{43}$$

Eqs. (41), (42) and (43) solve the problem.

13. To show that we are at liberty to make the assumptions expressed in (37) and (38), we observe that if we have any two real quantities, positive or negative, as x and y, we may put

$$x = m \sin M,$$
$$y = m \cos M,$$

as we then have

$$x^2 + y^2 = m^2 (\sin^2 M + \cos^2 M) = m$$

or $$m = \sqrt{x^2 + y^2}\,;$$

and also $$\frac{x}{y} = \frac{\sin M}{\cos M} = \tan M$$

or $$M = \tan^{-1}\frac{x}{y}.$$

These values of m and M are always real and possible, whatever be the values or signs of x and y, hence there are some real values of m and M which will satisfy (37) and (38).

14. Given the *right ascension* and *declination* of a body, and the *obliquity of the ecliptic*, to find the *longitude* and *latitude* of the body.

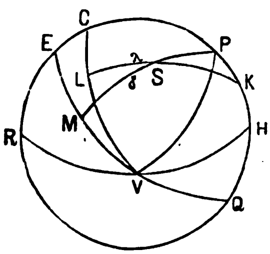

FIG. 3.

Let $HPER$, Fig. 3, be that position of the meridian which coincides with the solstitial colure, and which is therefore perpendicular to both the equator, EQ, and the ecliptic CV. The vernal equinox V is then in the horizon, PV is the equinoctial colure, and the arcs CV, EV and QV are quadrants. Let K be the pole of the ecliptic, then in the triangle KPS,

$KP = CE = \omega =$ obliquity of ecliptic;
$PS = 90° - \delta$; $KS = 90° - \lambda$;
angle $KPS =$ arc $QM = QV + VM = 90° + \alpha$;
angle $PKS =$ arc $CL = CV - VL = 90° - L$

15. In the first equations of (7), (8) and (9), making $A = 90° + \alpha$, $B = 90° - L$, $a = 90° - \lambda$, $b = 90° - \delta$, $c = \omega$, we obtain the following:

$$\cos\lambda \cos L = \cos\delta \cos\alpha \tag{44}$$
$$\sin\lambda = \sin\delta\cos\omega - \cos\delta\sin\omega\sin\alpha \tag{45}$$
$$\cos\lambda\sin L = \sin\omega\sin\delta + \cos\omega\cos\delta\sin\alpha \tag{46}$$

In order to put these equations in a form adapted to logarithms, make

$$\sin \delta = m \sin M \tag{47}$$

$$\cos \delta \sin \alpha = m \cos M \tag{48}$$

which reduce (45) and (46) to

$$\begin{aligned} \sin \lambda &= m\,(\sin M \cos \omega - \cos M \sin \omega) \\ &= m \sin (M - \omega) \end{aligned} \tag{49}$$

$$\begin{aligned} \cos \lambda \sin L &= m\,(\cos M \cos \omega + \sin M \sin \omega) \\ &= m \cos (M - \omega) \end{aligned} \tag{50}$$

Dividing (47) by (48),

$$\tan M = \frac{\tan \delta}{\sin \alpha} \tag{51}$$

Dividing (50) by (44),

$$\tan L = \frac{m}{\cos \delta} \cdot \frac{\cos M - \omega)}{\cos \alpha},$$

but by (48), $$\frac{m}{\cos \delta} = \frac{\sin \alpha}{\cos M},$$

whence $$\tan L = \frac{\tan \alpha \cos (M - \omega)}{\cos M} \tag{52}$$

Dividing (49) by (50),

$$\frac{\tan \lambda}{\sin L} = \tan (M - \omega),$$

whence $$\tan \lambda = \sin L \tan (M - \omega) \tag{53}$$

Eqs. (51), (52) and (53) solve the problem.

CHAPTER II.

TIME.

Conversion of Different Kinds of Time.

16. A *sidereal* day is the interval between two successive meridian passages of the vernal equinox. An *apparent solar* day is the interval between two successive meridian passages of the sun's center. A *mean solar* day is the average length of all the apparent solar days in a tropical year.

17. The tropical year contains 365.24 mean solar days. But since the sun makes an apparent revolution from west to east in the ecliptic in a year, it loses one diurnal revolution from east to west, in comparison with the fixed stars; hence there are just one more sidereal than solar days in a year, namely, 366.24.

Hence $\quad 365.24\ solar\ days = 366.24\ sid.\ days,$

and

$$1\ solar\ day = \frac{366.24}{365.24}\ sid.\ day = 1.0027379\ sid.\ day \tag{54}$$

or
$$1\ solar\ day = 1\ sid.\ day + 3^m\,56^s.555\ sid.\ time \tag{55}$$

Also, from (54),

$$1\ sid.\ day = 0.9972696\ solar\ day \tag{56}$$

or
$$1\ sid.\ day = 1\ solar\ day - 3^m\,55^s.909\ solar\ time \tag{57}$$

The excess of a mean solar above a sidereal day is therefore $3^m\,56^s.555$ of sidereal time, or $3^m\,55^s.909$ mean solar time.

18. *To convert intervals of mean solar time into equivalent intervals of sidereal time,* and *vice versa.*

Since a sidereal day, hour, etc., is shorter than a solar day, hour, etc., a given interval will contain more sidereal than solar days or hours, etc. Hence it follows from (54), that any interval expressed in mean solar time may be changed into its sidereal equivalent by multiplying by 1.0027379; and from (56), that any sidereal interval may be changed into its mean solar equivalent by multiplying by 0.9972696.

In practice, the equivalent intervals of mean solar and sidereal time may be taken directly from a table.

19. *Given the sidereal time at any instant to find the mean solar time.*

Let m = mean time at given instant,

s = sid. time at given instant,

s' = sid. time at preceding mean noon.

Then m = mean interval elapsed since mean noon,

and $s - s'$ = sid. interval elapsed since mean noon.

Hence $$m = \textit{mean equivalent of } (s - s') \qquad (58)$$

and $$s - s' = \textit{sid. equivalent of } (m) \qquad (59)$$

From (58) we have this rule:

From the given sidereal time subtract the sidereal time at preceding mean noon, and reduce the remainder to its mean equivalent.

Given the mean solar time at any instant, to find the sidereal time.

From (59) we find

$$s = s' + \textit{sid. equivalent of } (m) \qquad (60)$$

whence the rule:

To the sidereal time at preceding mean noon add the given mean time reduced to its sidereal equivalent.

20. *Given the mean solar time at any instant to find the apparent solar time, or given the apparent to find the mean solar time.*

The *equation of time* is a quantity which, being added to the apparent time, gives the mean time. Hence the mean time may be found from the apparent by adding the equation of time, that is, applying it according to its sign; and the apparent time may be found from the mean by applying the equation of time with a contrary sign.

The "sidereal time at mean noon" and "equation of time" may be found for any day from the solar ephemeris.

Hour Angles.

21. The *hour angle* of a celestial body is the angle which its hour circle makes with the meridian.

Thus, the hour angle of the star S, Fig. 1, is the angle ZPS, or the arc EM of the equator.

Like the right ascension, the hour angle, being measured by an arc of the equator, may be expressed in either degrees, minutes and seconds of arc, or in hours, minutes and seconds of time, allowing 15° to 1 hour.

The hour angle is always reckoned *westward* from the meridian, from 0° to 360°, or from 0^h to 24^h.

22. Since the sidereal day begins when the vernal equinox is on the meridian, it follows that *the sidereal time at a given instant is equal to the hour angle of the vernal equinox.*

If, then, the hour angle of a star be given, to find the sidereal time, we have, Fig. 1, $EV = EM + MV$, that is,

$$\textit{sid. time} = \textit{star's hour angle} + \textit{star's R. A.} \qquad (61)$$

in which, if the sum of the hour angle and R. A. should be greater than 24 hours, the excess will be the sidereal time required.

23. Conversely, if the sidereal time at any instant be given, to find the hour angle of a star, we have from (61),

$$\textit{star's hour angle} = \textit{sid. time} - \textit{star's R. A.} \qquad (62)$$

and if the sid. time be less than the R. A., it must be increased by 24 hours, to render the subtraction possible.

In applying (61) and (62), the hour angle, as well as right ascension, must be expressed in *time*.

24. Since the apparent solar day begins when the sun's center is on the meridian, it follows that *the apparent time at a given instant is equal to the sun's hour angle.*

Hence, if the sun's hour angle be given to find the mean solar time, we have, Art. 20,

$$\textit{mean time} = \textit{sun's hour angle} + \textit{eq. of time} \qquad (63)$$

and if the mean time be given, to find the sun's hour angle,

$$\textit{sun's hour angle} = \textit{mean time} - \textit{eq. of time} \qquad (64)$$

in which, of course, the sun's hour angle must be expressed in *time*.

25. If the "sid. time" in (62), and "mean time" in (64) are taken by observation from the clock, the *clock error* must be known and allowed for, otherwise the resulting hour angle will be incorrect. But a method of finding the hour angle of a body independently of the clock error, which can be employed in certain cases, is to observe the *interval* of the times when the body has equal altitudes east and west of the meridian. It is evident that half this interval expressed in arc is the hour anlge $ZPS = EM$, Fig. 1.

Time of Meridian Passage.

26. To find the time when a star or other celestial body will pass the meridian, we observe that at this time its hour angle is 0, hence (61) becomes

$$\textit{sid. time of meridian passage} = \textit{star's R. A.} \qquad (65)$$

which may be converted into mean time, if desired, by Art. 19.

To find the mean time of the sun's meridian passage, (63) gives in the same way,

mean time sun's meridian passage = *eq. of time.*

This result is expressed in astronomical time; if it be required in civil time, 12 hours must be added, thus,

$$\textit{mean time sun's merid. passage} = \textit{eq. of time} + 12^h \quad (66)$$

Methods of Finding the Time by Observation.

I.

27. *By observing the time of meridian passage of a star, or the sun.*

Compute by (65) or (66) the time when a star or the sun's center will pass the meridian; then observe the same with the Transit instrument and clock. The difference of the computed and observed times will be the *clock error.*

Stars situated near the equator are the best to use for this purpose, provided the Transit instrument be well adjusted, because their apparent diurnal motion is most rapid.

This is the most simple and accurate method of finding the time.

II.

28. *By observing equal altitudes of a star or the sun.*

If we observe the times when a star has equal altitudes east and west of the meridian, and take their mean, we get the time of its meridian passage. But if the sun be observed in like manner, a correction will be necessary on account of its change of declination during the interval. This correction is called the *Equation of Equal Altitudes of the Sun.*

29. To find what effect the change of declination has on the sun's hour angle, take the general equation (14),

$$\sin h = \sin \delta \sin \phi + \cos \delta \cos \phi \cos P,$$

and differentiate it on the supposition that δ and P are the only variables; we find

$$0 = \sin \phi \cos \delta\, d\delta - \cos \delta \cos \phi \sin P dP - \cos \phi \cos P \sin \delta\, d\delta,$$

whence

$$d\delta\,(\sin \phi \cos \delta - \cos \phi \sin \delta \cos P) = \cos \phi \cos \delta \sin P dP,$$

and

$$dP = d\delta \left(\frac{\sin \phi}{\cos \phi \sin P} - \frac{\sin \delta \cos P}{\cos \delta \sin P}\right)$$

$$= d\delta \left(\frac{\tan \phi}{\sin P} - \frac{\tan \delta}{\tan P}\right) \qquad (67)$$

30. Let t = half the interval between the observations, expressed in hours;

Δ = sun's hourly change of declination;

x = correction to be added to the mean of the observed times, to obtain apparent noon.

Then $t\Delta$ will be the sun's change of declination during half the interval, and x will be the change it produces in the hour angle. These quantities are very small, and have sensibly the same relation as $d\delta$ and dP, hence we may substitute in Eq. (67), $t\Delta$ for $d\delta$, t for P, and x for dP.

If the declination is increasing, the hour angle west of the meridian will be greater than that east of the meridian, hence the mean of the observed times will be *after* apparent noon, and will require a negative correction. That is, Δ *and* x *have contrary signs.*

31. Making the substitutions referred to above, changing the sign of the second member, and dividing by 15 to get the correction in *time,* we have

$$x = -\frac{t\Delta}{15}\left(\frac{\tan \phi}{\sin t} - \frac{\tan \delta}{\tan t}\right) \qquad (68)$$

If we put $\frac{-t}{15 \sin t} = A$, and $\frac{t}{15 \tan t} = B$,

(68) becomes

$$x = A\Delta \tan \phi + B\Delta \tan \delta \tag{69}$$

the equation of equal altitudes.

The logarithms of A and B are computed for different values of the interval $2t$, and may be taken from a table. Such a table is given in CHAUVENET'S "Spherical and Practical Astronomy," Vol. II.

The values of δ and Δ may be taken from the Nautical Almanac, observing that δ is negative when the declination is south, and Δ is negative when the declination is decreasing.

32. The equal altitudes may be measured with the sextant. As these altitudes themselves are not used, no corrections for parallax, refraction, etc., are required.

The mean of the observed times corrected for the equation of equal altitudes is the *observed* time of apparent noon. Comparing this with the *mean* time of apparent noon, found by (66), we get the *clock error.*

III.

33. *By measuring the altitude of a body out of the meridian.*

Equation (32) may be written in the form

$$\sin \tfrac{1}{2} P = \sqrt{\frac{\sin \frac{1}{2}[z - (\phi - \delta)] \sin \frac{1}{2}[z + (\phi - \delta)]}{\cos \delta \cos \phi}} \tag{70}$$

In this equation, the latitude of the place, (ϕ), is supposed to be known, the zenith distance of the body, (z), is the complement of the measured altitude, and the declination, (δ), is to be taken from the Nautical Almanac, observing that north declinations are to be marked $+$, and south $-$.

The altitude may be measured with the sextant, and must

be corrected in all cases for *refraction*, and in the case of the sun or a planet, for *parallax* and *semidiameter*. The *time* of the observation must also be observed by the clock.

By applying logarithms to Eq. (70) we obtain the value of P, the hour angle, which is to be changed into time by (61) or (63). Comparing this with the *observed* time we have the *clock error*.

The best time for measuring the altitude is when the body is near the prime vertical, its motion being then most rapid.

Time of Rising and Setting.

34. To find the time of *true* rising or setting, we have only to take Eq. (22) for the hour angle of a body in the horizon, that is,

$$\cos P = - \tan \phi \tan \delta,$$

and change the hour angle P into time, by (61) in case of a star, or (63) in case of the sun.

35. To find the time of *apparent* rising or setting, that is, the time when a body appears in the horizon, we use Eq. (70), making

$$z = 90^\circ + \text{refraction} = 90^\circ\ 34\tfrac{1}{2}',$$

since the body at this time is really $34\tfrac{1}{2}'$ below the horizon.

If the time of apparent rising or setting of the *sun's upper limb* be required, it is necessary to make in Eq. (70),

$$z = 90^\circ + \text{refraction} + \text{semidiameter} = 90^\circ\ 50' \text{ nearly.}$$

CHAPTER III.

THE MERIDIAN LINE.

36. The intersection of the plane of the meridian at any place with that of the horizon is called the *meridian line.* The following are some of the best methods of finding the direction of the meridian line at a given place.

I.

37. *By observing the azimuth of a circumpolar star at the time of its greatest elongation.*

A star is said to be at its greatest elongation when its vertical circle ZS, Fig. 1, is tangent to its diurnal circle, that is, perpendicular to its hour circle PS. Hence in this position the parallactic angle $S = 90°$, and $\cos S = 0$, $\sin S = 1$, whence by (12),

$$\sin Z = \frac{\cos \delta}{\cos \phi} \tag{71}$$

We also find from (16),

$$\sin h = \frac{\sin \phi}{\sin \delta} \tag{72}$$

and from (19),
$$\cos P = \frac{\sin h \cos \delta}{\cos \phi},$$

or by (72),
$$\cos P = \frac{\sin \phi \cos \delta}{\cos \phi \sin \delta} = \frac{\tan \phi}{\tan \delta} \tag{73}$$

The latitude of the place being known, and the declination of the star being taken from the Nautical Almanac, the values of Z and P may be found from (71) and (73).

The hour angle P must then be changed into time by (61), observing that if the time of greatest *eastern* elongation be required, P must be taken with a negative sign.

The sidereal time of greatest elongation thus obtained may be converted into mean time, if desired, by Art. 19.

38. A little before this time, direct to the star an instrument provided with a horizontal circle for measuring azimuths, such as the engineer's transit, or theodolite, and let the motion of the star be followed by the instrument until it ceases to change its azimuth, then take the reading of the circle. If this reading be denoted by R, the reading for the meridian will be $R \pm Z$.

Thus, suppose the readings of the instrument to increase from the left round to the right, then if the observed elongation be *west*, the reading for the meridian is $R + Z$, but if *east*, it is $R - Z$. The reverse will be true if the circle reads from the right round to the left.

39. If the latitude of the place be not known, the reading of the instrument may be taken when the star is at both greatest eastern and greatest western elongation, and the mean of the two will be the reading for the meridian.

40. The star selected should be very near the pole. On account of its large magnitude and close proximity to the pole, the pole-star is well adapted to this purpose.

II.

41. *By measuring the azimuth and altitude of a body at the same instant.*

Eq. (33) may be written thus:

$$\sin \tfrac{1}{2} Z = \sqrt{\frac{\sin \frac{1}{2}(z + \phi - \delta) \cos \frac{1}{2}(z + \phi + \delta)}{\cos \phi \sin z}} \qquad (74)$$

in which z is the complement of the observed altitude corrected for refraction, etc., and Z is the azimuth reckoned from the north point. If then R denote the observed reading of the horizontal circle, $R \pm Z$ will be its reading for the meridian.

42. The instrument employed should have both a horizontal and a vertical circle, for measuring azimuths and altitudes, as in the theodolite. But if it is adapted for measuring azimuths only, the altitude may be measured at the same instant by another observer with the sextant.

III.

43. *By observing the azimuth of a body, and the corresponding time.*

From the observed time, the hour angle of the body, P, may be found by Eq. (62) or (64); then by (42),

$$\tan Z' = \frac{\cos M \tan P}{\sin (\phi - M)} \tag{75}$$

in which M is determined by (41),

$$\tan M = \frac{\tan \delta}{\cos P} \tag{76}$$

The value of Z' is the azimuth reckoned from the south point. If R denote the observed azimuth reoding, $R \pm Z'$ will be the reading for the meridian.

44. It may be remarked that if the magnetic bearing of the body be taken, the difference between this and the true azimuth Z' will be the "variation of the needle."

IV.

45. *By observing equal altitudes of a star or the sun.*

If the instrument be directed to a star when at equal altitudes east and west of the meridian, and the readings of

the horizontal circle taken in the two positions, their mean will be the reading for the meridian.

But if the *sun* be observed at equal altitudes, the mean of the two readings will require a correction on account of the change of the sun's declination. To find this correction, take the general equation (15),

$$\sin \delta = \sin \phi \sin h + \cos \phi \cos h \cos Z,$$

and replace Z by $180° - Z'$, Eq. (6), since the sun's azimuth is more conveniently reckoned from the south point. We thus find

$$\sin \delta = \sin \phi \sin h - \cos \phi \cos h \cos Z' \qquad (77)$$

To find what effect the change of declination produces in the azimuth, differentiate (77) on the supposition that δ and Z' are the only variables, then

$$\cos \delta \, d\delta = \cos \phi \cos h \sin Z' dZ',$$

whence
$$dZ' = d\delta \frac{\cos \delta}{\cos \phi \cos h \sin Z'} \qquad (78)$$

But from (11), since $\sin Z = \sin Z'$,

$$\cos \delta = \frac{\cos h \sin Z'}{\sin P},$$

which reduces (78) to

$$dZ' = \frac{d\delta}{\cos \phi \sin P} \qquad (79)$$

46. Let the times of the two observations be noted by the clock; let $\triangle = \frac{1}{2}$ the sun's change of declination in the interval, and $x =$ correction required by the mean of the two azimuth readings; then x and $\triangle$ will have sensibly the same relation as dZ' and $d\delta$, hence by (79),

$$x = \frac{\triangle}{\cos \phi \sin P} \qquad (80)$$

The change of declination,, $\triangle$, may be found by means of the hourly change given in the Nautical Almanac, and the hour angle P is half the interval of *apparent* time

between the observations, converted into arc by multiplying by 15. (Art. 25.)

47. In regard to the sign of the correction x, it is evident that if the declination is *increasing*, the effect is to cause the mean of the two observed azimuth readings to lie on the *west* of the meridian, and if *decreasing*, on the *east* of the meridian. It follows that if the horizontal circle reads from the left round to the right, x has a contrary sign to $\triangle$, but if from right round to left, x has the same sign as $\triangle$.

V.

48. *By observing the meridian transits of two stars at very different altitudes.*

To adjust the transit instrument nearly in the meridian.

Let the instrument be carefully leveled; it will then describe a vertical circle which at the zenith coincides with the meridian, and departs from it more and more in approaching the horizon. The observed time of transit of a star near the zenith will therefore be nearly the same as the time of its meridian transit, however much the instrument may deviate; but for a star near the horizon, the time of observed transit will differ considerably from that of meridian transit.

If, then, by observing the times of transit of a high and a low star, the difference of observed sidereal times is found equal to the difference of their right ascensions—that is, both stars give the same error for the clock—it will follow that the instrument is adjusted in the meridian, but if not, that it deviates in azimuth. The time of observed transit of the higher star may be assumed to be correct; hence if the lower star was observed too early, the deviation is east, but if too late, it is west.

Let the error be corrected as nearly as may be, and the observation repeated until the high and low stars give

nearly the same clock error; the instrument will then be approximately adjusted in the meridian.

49. *To find the amount of the deviation in azimuth.*

We may distinguish several cases according to the position of the two stars at the time of their culmination.

Case 1.—Suppose both stars *south of the zenith.*

Since we have, Eq. (6), $\sin Z' = \sin Z$, the general equation (11) may be written

$$\cos h \sin Z' = \cos \delta \sin P.$$

Now as the instrument is supposed to be nearly in the meridian, the angles Z' and P are very small, and if they are expressed in seconds we may put for $\sin Z'$, $Z' \sin 1''$, and for $\sin P$, $P \sin 1''$; whence we find

$$P = Z' \frac{\cos h}{\cos \delta} \tag{81}$$

But for a star south of the zenith, we have by (2) and (25),

$$\cos h = \sin z = \sin (\phi - \delta).$$

Substituting this in (81), and dividing by 15 in order to find the hour angle P in *time,*

$$P = \frac{Z'}{15} \cdot \frac{\sin (\phi - \delta)}{\cos \delta} = \frac{Z'}{15} (\sin \phi - \cos \phi \tan \delta).$$

Now, for one of the stars, let

$t =$ sidereal time of observed transit,
$\alpha =$ " meridian transit
$=$ the star's right ascension [Eq. (65)],

then, by Eq. (61),

$$t = \alpha + P = \alpha + \frac{Z'}{15} (\sin \phi - \cos \phi \tan \delta),$$

and for the other star,

$$t' = \alpha' + P' = \alpha' + \frac{Z'}{15} (\sin \phi - \cos \phi \tan \delta'),$$

since Z', the azimuth deviation of the instrument, is the same for both stars. Hence we find

$$t - t' = \alpha - \alpha' - \frac{Z'}{15} \cos\phi\,(\tan\delta - \tan\delta')$$

and
$$Z' = \frac{15\,\{\alpha - \alpha' - (t - t')\}}{\cos\phi\,(\tan\delta - \tan\delta')} \tag{82}$$

which may be adapted to logarithms by the trigonometrical formula

$$\tan x - \tan y = \frac{\sin(x - y)}{\cos x \cos y},$$

then
$$Z' = \frac{15\,\{\alpha - \alpha' - (t - t')\} \cos\delta \cos\delta'}{\cos\phi \sin(\delta - \delta')} \tag{83}$$

It appears from (81) that P and Z' always have the same sign, hence a *positive* value of Z' in (83) indicates deviation *west of south*, and a *negative* value of Z', *east of south*. (Art. 21.)

Case 2.—Suppose both stars *north of the zenith and above the pole.*

In this case we have instead of (81),

$$P = Z \frac{\cos h}{\cos\delta} \tag{84}$$

and since the stars are between the zenith and the pole, (26) gives

$$\cos h = \sin(\delta - \phi),$$

whence, finding P in *time*,

$$P = \frac{Z}{15} \cdot \frac{\sin(\delta - \phi)}{\cos\delta} = \frac{Z}{15} \cos\phi \tan\delta - \sin\phi).$$

We then have, as before,

$$t = \alpha + \frac{Z}{15} (\cos\phi \tan\delta - \sin\phi),$$

$$t' = \alpha' + \frac{Z}{15} (\cos\phi \tan\delta' - \sin\phi),$$

$$t - t' = \alpha - \alpha' + \frac{Z}{15} \cos\phi\,(\tan\delta - \tan\delta');$$

hence $$Z = \frac{15 \{t - t' - (\alpha - \alpha')\}}{\cos \phi \, (\tan \delta - \tan \delta')} \tag{85}$$

which may be adapted to logarithms in the same way as (82).

It appears from (84) that P and Z always have the same sign, hence a *positive* value of Z in (85) shows the deviation to be *west of north*, and a *negative* value of Z, *east of north*.

Case 3.—Suppose both stars *north of the zenith, one above the pole, and the other below.*

For the higher star, we have, as before,

$$P = \frac{Z}{15} (\cos \phi \tan \delta - \sin \phi)$$

and $$t = \alpha + \frac{Z}{15} (\cos \phi \tan \delta - \sin \phi).$$

For the star below the pole, by (29),

$$\cos h = \sin \{180° - (\phi + \delta')\} = \sin (\phi + \delta'),$$

whence, by (84),

$$P' = \frac{Z}{15} \cdot \frac{\sin (\phi + \delta')}{\cos \delta'} = \frac{Z}{15} (\sin \phi + \cos \phi \tan \delta').$$

Now the time of passing the meridian below the pole $= \alpha' - 12$ hours, hence the sidereal time of observed transit

$$= t' = \alpha' - 12^h - P' = \alpha' - 12^h - \frac{Z}{15} (\sin \phi + \cos \phi \tan \delta').$$

Hence we find

$$t - t' = \alpha - \alpha' + 12^h + \frac{Z}{15} \cos \phi \, (\tan \delta + \tan \delta'),$$

and $$Z = \frac{15 \{t - t' - (\alpha - \alpha') - 12^h\}}{\cos \phi \, (\tan \delta + \tan \delta')} \tag{86}$$

or since $$\tan \delta + \tan \delta' = \frac{\sin (\delta + \delta')}{\cos \delta \cos \delta'},$$

$$Z = \frac{15 \{t - t' - (\alpha - \alpha') - 12^h\} \cos \delta \cos \delta'}{\cos \phi \sin (\delta + \delta')} \tag{87}$$

Case 4.—Suppose ***the same star*** to be observed ***both above and below the pole.***

We shall then have, in (86), $\alpha = \alpha'$, and $\delta = \delta'$, whence

$$Z = \frac{15\ \{(t - t') - 12^h\}}{2 \cos \phi \tan \delta} \tag{88}$$

CHAPTER IV.

LATITUDE.

50. The latitude of a place, (ϕ), considered astronomically, is the arc of the meridian ZE, Fig. 1, included between the zenith of the place and the equator. The colatitude, (ψ), is the arc ZP between the zenith and the pole. Now since ZH, ZR and PE are quadrants, we have $ZE = PH$, and $ZP = ER$; that is, the altitude of the pole above the horizon is equal to the latitude, and the altitude of the equator is equal to the colatitude.

The following are some of the principal methods of determining the latitude of a place.

I.

51. *By measuring the meridian altitude or zenith distance of a heavenly body.*

There may be three cases.

(1.) If the body culminates *south of the zenith,* (24) and (25) give

$$\psi = h - \delta \tag{89}$$

$$\phi = z + \delta \tag{90}$$

(2.) If the body culminates *north of the zenith, above the pole,* (26) and (27) give

$$\phi = h - p \tag{91}$$

$$\phi = \delta - z \tag{92}$$

(3.) If the body culminates *north of the zenith, below the pole,* (28) and (29) give

$$\phi = h + p \tag{93}$$

$$\phi = 180° - (\delta + z) \tag{94}$$

In applying these formulæ, the declination, δ, or polar distance, p, is found from the Nautical Almanac, and the meridian altitude, h, or zenith distance, z, by measurement, corrected for refraction, and in case of the sun, moon or a planet, for parallax and semidiameter.

II.

52. *By measuring the greatest and least altitudes of a circumpolar star.*

Let h' denote the greatest, and h'' the least altitude of the star; then by (91) and (93),

$$\left.\begin{aligned} \phi &= h' - p \\ \phi &= h'' + p \end{aligned}\right\}$$

half the sum of which is

$$\phi = \tfrac{1}{2}(h' + h'') \tag{95}$$

The measured altitudes must be corrected for refraction.

III.

53. *By measuring the meridian zenith distances of two stars on opposite sides of the zenith.*

Suppose the two stars to culminate at *nearly equal distances* north and south of the zenith, then by (90) and (92),

$$\left.\begin{aligned} \phi &= \delta + z \\ \phi &= \delta' - z' \end{aligned}\right\}$$

half the sum of which is

$$\phi = \tfrac{1}{2}(\delta + \delta') + \tfrac{1}{2}(z - z') \tag{96}$$

The instrument generally used in the application of this method is the *zenith telescope.* The two stars are so selected

that the interval between their culminations is sufficient to give time for turning the telescope 180° about a vertical axis, the inclination of the telescope to this axis being unchanged during the revolution. Hence the two stars, having nearly equal zenith distances, appear successively in the field of view, and the difference of their zenith distances can be accurately measured with a micrometer. The instrumental errors to which all other methods of finding the latitude are liable, are thus avoided.

Suppose m and m' to be the readings of the micrometer on the two stars, and R the value of one revolution of the micrometer screw, in seconds. Then, supposing the readings to increase with the zenith distances, the observed difference of zenith distance of the two stars in seconds will be $R\ (m - m')$.

54. To guard against any change of position in the vertical axis, during the observations, a delicate level is attached to the telescope, and read after the culmination of each star. Let n and s denote the readings of the north and south ends at the culmination of one star, and n' and s' at that of the other star, and d the value of one division of the level scale in seconds, then

$$d\,\frac{n - s}{2} \qquad \text{and} \qquad d\,\frac{n' - s'}{2}$$

will be the corresponding inclinations of the level to the horizon. Since any change of level which increases the apparent zenith distance of one star diminishes that of the other, the observed difference of zenith distance must be corrected by the *sum* of these inclinations, that is,

$$\frac{d}{2}\,(n - s + n' - s').$$

A correction is also required for the difference of refraction due to the slight difference of zenith distance of the two stars. Let r and r' be the two refractions, then as they

diminish the apparent zenith distance of both stars, their *difference*, $r - r'$, will be the correction to be added. Hence the corrected difference of zenith distance is

$$z - z' = R\,(m - m') + \frac{d}{2}\,[n + n' - (s + s')] + r - r',$$

and Eq. (96) becomes

$$\phi = \tfrac{1}{2}\,(\delta + \delta') + \frac{R}{2}\,(m - m') + \frac{d}{4}\,[n + n' - (s + s')] + \tfrac{1}{2}\,(r - r') \qquad (96)a$$

If the observation on either star was made after its meridian passage, another correction must be added for the "reduction to the meridian." See Art. 62.

55. This method of finding the latitude excels all others, both in simplicity and in accuracy. It is the invention of Capt. ANDREW TALCOTT, late of the U. S. Corps of Engineers.

IV.

56. *By observing the transits of a star over the prime vertical.*

When a body is on the prime vertical, its azimuth $Z = 90°$, hence $\cos Z = 1$, and (15) becomes

$$\sin\delta = \sin\phi\sin h$$

whence

$$\sin h = \frac{\sin\delta}{\sin\phi} \qquad (97)$$

and (17) becomes

$$\cos P = \frac{\sin\delta\cos\phi}{\cos\delta\sin\phi} = \frac{\tan\delta}{\tan\phi} \qquad (98)$$

whence

$$\tan\phi = \frac{\tan\delta}{\cos P} \qquad (99)$$

From (99) we may find the latitude, ϕ, when we know δ, the star's declination, and P, its hour angle on the prime vertical. The value of δ is found from the Nautical Almanac, and that of P by observation, as follows: Any star

whose declination is north, and less than the latitude of the place, will cross the meridian between the zenith and equator, and will cross the prime vertical at two points equidistant from the meridian, and on opposite sides of it. Having a transit instrument adjusted with its axis north and south, so that the telescope revolves in the plane of the prime vertical, we observe the passage of such a star over the wires of the instrument, at both positions where it crosses the prime vertical. The sidereal interval between the observations is evidently double the hour angle of the star on the prime vertical; Art. 25. Hence the value of P in (99) is half the observed sidereal interval converted into arc by multiplying by 15.

57. The Transit instrument may be approximately adjusted in the prime vertical by placing it on a star at the instant of passing the prime vertical, as nearly as the time can be ascertained. For this purpose, P may be computed from (98), using the best value of ϕ which can be obtained, then the required time may be found by (61), and the star's altitude at the same moment is given by (97).

If the Transit is accurately adjusted in the prime vertical, the mean of the observed sidereal times, corrected for the error of the clock, should be equal to the star's right ascension; hence if it is not so, the difference will measure the azimuth error of the instrument. Any such error, in either direction, makes the resulting latitude too great.

58. The telescope should be reversed on its supports between the observations, and the east and west transits observed with the axis in reversed positions. All the errors of adjustment, except the level error, will thus be eliminated. The level error should be well determined, and applied to the result.

The observations on each wire must be reduced to the middle wire by means of the equatorial intervals, or the latitude determined from the observations on each wire separately, and the mean of the results taken for the latitude.

V.

59. *By observing the altitude and hour angle of a body at the same instant.*

We have the general equation (14),

$$\sin h = \sin \delta \sin \phi + \cos \delta \cos \phi \cos P \tag{100}$$

in which the value of δ may be found from the Nautical Almanac, and those of h and P by observation; hence ϕ is the only unknown quantity. To determine it, let us assume m and M such as to satisfy the conditions,

$$\sin \delta = m \sin M \tag{101}$$

$$\cos \delta \cos P = m \cos M \tag{102}$$

whence

$$\tan M = \frac{\tan \delta}{\cos P} \tag{103}$$

Equation (100) then becomes

$$\sin h = m (\sin \phi \sin M + \cos \phi \cos M) = m \cos (\phi - M),$$

whence

$$\cos (\phi - M) = \frac{\sin h}{m},$$

or from (101),

$$\cos (\phi - M) = \frac{\sin h \sin M}{\sin \delta} \tag{104}$$

Equation (103) makes known M, and (104) $\phi - M$, and consequently ϕ, the latitude required.

60. The altitude, h, may be measured with the sextant, corrected for refraction, and if of the sun, parallax and semidiameter. The hour angle, P, may be found by observing the time at which the altitude is measured, and substituting it in (62) or (64). The *clock error* must be accurately

known, however, as any error in the observed time will affect the value of P.

But the hour angle may be found independently of the clock error, as explained in Art. 25, by observing the times of equal altitude east and west of the meridian. Half the difference of *sidereal* times of equal altitudes of a star, or half the difference of *apparent* times of those of the sun, is the hour angle expressed in time, the clock error being eliminated in taking the interval. The clock *rate* must be known, however, as it affects the observed interval.

If the hour angle is found by this method, the value of δ to be used in (103) and (104) should be the mean of the declinations for the two observed times.

VI.

61. *By observing circum-meridian altitudes of a body, and the corresponding times.*

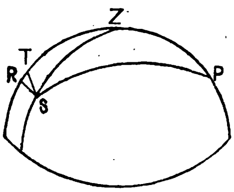

FIG. 4.

This method consists in measuring several altitudes of a body just before and just after its meridian passage, applying to each a correction called the "reduction to the meridian," and taking their mean as the meridian altitude, from which the latitude may be found by the 1st method.

62. The *reduction to the meridian* is the difference of any observed altitude or zenith distance, and the *meridian* altitude or zenith distance.

Let ST, Fig. 4, be an arc of the diurnal circle of the body S, and SR an arc parallel to the horizon;

let $\quad z = ZS = ZR,$

and $\quad x = TR = ZS - ZT =$ reduction to the meridian.

By Eq. (25), $\qquad ZT = \phi - \delta,$

hence $x = z - (\phi - \delta)$, or $z = x + (\phi - \delta)$,

and $\cos z = \cos x \cos (\phi - \delta) - \sin x \sin (\phi - \delta)$.

But the observations being made near the meridian, say within 10 minutes of the time of meridian passage, x is very small, and we may put

$$\cos x = 1, \qquad \sin x = x \sin 1'',$$

also we may replace $\cos z$ by $\sin h$, then

$$\sin h = \cos (\phi - \delta) - x \sin (\phi - \delta) \sin 1'' \qquad (105)$$

If now we substitute in (14), $\cos P = 1 - 2 \sin^2 \tfrac{1}{2} P$, it becomes

$$\sin h = \sin \delta \sin \phi + \cos \delta \cos \phi - 2 \cos \delta \cos \phi \sin^2 \tfrac{1}{2} P,$$

or $\sin h = \cos (\phi - \delta) - 2 \cos \delta \cos \phi \sin^2 \tfrac{1}{2} P$ (106)

Equating the second members of (105) and (106), we find

$$x \sin (\phi - \delta) \sin 1'' = 2 \cos \delta \cos \phi \sin^2 \tfrac{1}{2} P,$$

whence $$x = \frac{2 \sin^2 \tfrac{1}{2} P}{\sin 1''} \cdot \frac{\cos \delta \cos \phi}{\sin (\phi - \delta)},$$

or $$x = k \cdot \frac{\cos \delta \cos \phi}{\sin (\phi - \delta)} \qquad (107)$$

in which x is expressed in *seconds*. This is the correction to be applied to the observed altitudes.

The values of

$$k = \frac{2 \sin^2 \tfrac{1}{2} P}{\sin 1''}$$

are computed for values of P to every second, and may be taken from a table.

Since Eq. (107) involves the latitude, ϕ, it must be approximately known before this method can be used.

63. If the clock has a sensible *rate* during the observations, it must be taken account of.

Let r = daily rate, positive when *losing*; then if P is the hour angle indicated by the clock, and P' the true hour angle, we have

$$P' : P :: 24^h : 24^h - r :: 86400^s : 86400^s - r,$$

whence
$$\frac{P'}{P} = \frac{86400}{86400 - r} = \frac{1}{1 - \dfrac{r}{86400}}$$

Instead of sin $\frac{1}{2}$ P, we must use sin $\frac{1}{2}$ P', but since P is always small,

$$\sin \tfrac{1}{2} P' = \sin \tfrac{1}{2} P \cdot \frac{P'}{P} \text{ nearly, and}$$

$$\sin^2 \tfrac{1}{2} P' = \sin^2 \tfrac{1}{2} P \left(\frac{P'}{P}\right)^2 = \sin^2 \tfrac{1}{2} P \left[\frac{1}{1 - \dfrac{r}{86400}}\right]^2$$

which may be written $n \cdot \sin^2 \frac{1}{2} P$. The logarithm of n may be taken from a table, and the factor k in (107) now becomes nk.

CHAPTER V.

LONGITUDE.

64. The *difference of longitude* of two places on the earth's surface is the arc of the celestial equator included between their meridians. If the meridian of one of the places be assumed as the *first meridian*, their difference of longitude is called *the longitude* of the other place.

Longitude may be expressed in either degrees, minutes and seconds of arc, or hours, minutes and seconds of time, 15 degrees being equal to 1 hour.

65. Nearly all methods of finding differences of longitude depend on the principle that at any given instant, *the difference of local time at any two places is the same as their difference of longitude expressed in time.*

The method of determining the longitude from the observation of a *solar eclipse*, is given in Chapter VIII. The method by *chronometers*, and that by the *telegraph*, are explained in Descriptive Astronomy. The latter method, applied to the transmission of "star signals" by telegraph, is incomparably the best of all known methods, wherever practicable, both on account of the great precision with which the operation can be conducted, and of the facility with which it may be repeated indefinitely. The result, moreover, is independent of the clock error, and of the errors of the star tables.

We now proceed to explain another method of determining longitude, very simple in practice, and capable, with proper precautions, of giving results with considerable accuracy.

66. *Method of finding the longitude of a place by observing a Moon-culmination.*

The moon passes the meridian of a given place about 50 minutes later on any day than on the preceding day, hence its increase of right ascension is about 50^m in 24 hours, or about 2^m an hour. Hence if the moon and a certain star should pass the meridian of the given place at the same moment, the times of their passing the meridian 1 hour or 15° west of this would differ about 2^m, the times of their passing the meridian 2^h or 30° west would differ about 4^m, and so on. If then at any two places the interval between the culminations of the moon and a given star be observed, the difference of those intervals, divided by the moon's hourly change of R. A., will give the difference of longitude of the two places, expressed in hours.

67. If the observations were made at only one place, the longitude of that place may still be found by comparing the moon's R. A., at the time of its observed culmination, with its R. A. at the time of passing the *first meridian*, found from the Nautical Almanac.

The observer at the given place notes the times of meridian passage of the moon's bright limb and that of a star whose R. A. is known; the interval expressed in sidereal time will be their difference of R. A., from which the R. A. of the moon's limb will become known. The R. A. of the moon's center is then found by adding to that of the first limb, or subtracting from that of the second, the sidereal time required for the semidiameter to pass the meridian.

68. Now, let

α = R. A. of moon's center at time of transit,

or, by Eq. (65),

α = local sidereal time at instant of transit,

and let

S = Greenwich sidereal time at same instant,

then
$$\textit{Longitude} = S - \alpha \tag{108}$$

69. It remains to show how the value of S may be found by means of the Lunar Ephemeris.

Let

α_1 = R. A. next less than α, given in the Ephemeris,

M = Greenwich mean time corresponding to α,

M_1 = Greenwich mean time corresponding to α_1,

then

$\alpha - \alpha_1$ = change of R. A. in the interval $M - M_1$;

let

$\triangle$ = increase of moon's R. A. in 1 minute at the time M_1, expressed in seconds,

and

ε = hourly increase of $\triangle$.

Now the second difference (ε) will be found to be sensibly constant, hence the first difference ($\triangle$) varies uniformly, and *its mean value for any interval is the value which it has at the middle of that interval.* Let the interval ($M - M_1$) be expressed in seconds. then the mean value of $\triangle$ for this interval will be

$$\triangle + \tfrac{1}{2}(M - M_1)\,\frac{\varepsilon}{3600}.$$

The increase of R. A. in the interval ($M - M_1$), divided by its increase in 1 minute, gives the number of minutes in the interval; that is,

$$\frac{\alpha - \alpha_1}{\Delta + \frac{1}{2}(M - M_1)\frac{\varepsilon}{3600}} = \frac{M - M_1}{60}.$$

Let $M - M_1 = x$, then

$$x = \frac{60(\alpha - \alpha_1)}{\Delta\left(1 + \frac{x}{7200}\cdot\frac{\varepsilon}{\Delta}\right)} = \frac{60(\alpha - \alpha_1)}{\Delta}\left(1 - \frac{x}{7200}\cdot\frac{\varepsilon}{\Delta}\right),$$

nearly; and putting

$$\left.\begin{aligned} \frac{60(\alpha - \alpha_1)}{\Delta} &= x' \\ \frac{x'^2}{7200}\cdot\frac{\varepsilon}{\Delta} &= x'' \end{aligned}\right\} \qquad (109)$$

since x' is an approximate value of x, we have

$$x = x' - x'',$$

or

$$M = M_1 + x' - x'' \qquad (110)$$

This value of M being converted into sidereal time by Art. 19, gives S to be used in Eq. (108).

70. The stars observed in connection with the moon should be situated near the moon, as in that case the errors of the Transit instrument and clock will not affect the result, for being the same for both moon and star, they will be eliminated in finding the observed interval. Such stars are called *Moon-culminating Stars.*

CHAPTER VI.

THE METHOD OF LEAST SQUARES.

71. The ***Method of Least Squares*** is a mathematical process based on the Calculus of Probabilities, and chiefly used in determining the best methods of combining the data of Practical Astronomy, and in discussing the accuracy of the results. The first principles of this method we now proceed to explain, beginning with one or two fundamental doctrines of Probability. It will be convenient in this Chapter to make some changes in the notation adopted at the beginning, and which has been followed hitherto.

Theory of Probabilities.

72. The mathematical ***probability*** of any event is the ratio of the number of ways in which it may happen, to the whole number of ways in which it may either happen or fail. For instance, if we have a box containing 9 white balls and 1 black ball, and if a person blindfold proceed to draw a ball from the box, the probability of his drawing a white ball will be $\frac{9}{10}$, and that of his drawing a black ball will be $\frac{1}{10}$.

In general, suppose the box to contain a white and b black balls, then,

$$\text{probability of drawing white ball} = \frac{a}{a+b},$$

$$\text{“} \qquad \text{“} \quad \text{black} \text{ “} = \frac{b}{a+b}.$$

But $$\frac{a}{a+b} + \frac{b}{a+b} = \frac{a+b}{a+b} = 1,$$

and since either a white or black ball must be drawn, the sum of their probabilities is certainty, hence

Unity is the measure of certainty.

73. Suppose now another box containing a' white and b' black balls, and let

p = probability of drawing white ball from 1st,
p' = " " " " " 2d,
P = " " " " " both.

Then $$p = \frac{a}{a+b}, \qquad p' = \frac{a'}{a'+b'},$$

and to find P we have

number of possible cases $= (a + b)(a' + b')$,
" favorable " $= aa'$,

hence $$P = \frac{aa'}{(a+b)(a'+b')} = pp' \tag{111}$$

which shows that ***the probability of any compound event, made up of two independent events, is the product of their separate probabilities.*** The same may be shown to hold true of the combination of ***any number*** of simple and independent events.

Probability of Errors of Observation.

74. The most important application of the Theory of Probabilities is to deducing the ***most probable*** results from many observations, each of which is necessarily more or less imperfect. It should be remarked, however, that trustworthy results can only be obtained when the number of observations under discussion is very great, and it must be premised at the outset, that the rules and formulæ deduced in this Chapter hold good only in the case of large numbers of observations.

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75. Let there be a series of m observations of some quantity x, giving the results n, n', n'', &c., and affected with the errors $\Delta, \Delta', \Delta''$, &c., then their *mean* will be

$$x = \frac{n + n' + n'' + \&c.}{m} \tag{112}$$

Now it is safe to assume that the mean of several equally good observations is the most probable value of the quantity observed, or, in other words, that in similar circumstances, *positive and negative errors of equal magnitude are equally probable.* According to this principle, we have

$$\Delta = n - x, \quad \Delta' = n' - x, \quad \Delta'' = n'' - x, \&c. \tag{113}$$

We also have from (112),

$$n + n' + n'' + \&c. = mx = x + x + x + \&c.,$$

hence

$$(n - x) + (n' - x) + (n'' - x) + \&c. = 0,$$

or by (113),

$$\Delta + \Delta' + \Delta'' + \&c. = 0 \tag{114}$$

Now a small error would be more likely to be committed than a large one, hence the probability diminishes as the error increases; that is, the probability is a function of the error itself.

Let $\phi\Delta$ = probability of the error Δ,
$\phi\Delta'$ = " " " Δ',
&c. &c.

If in the series of m observations, the error Δ occurs a times, the error Δ', a' times, &c., then, by the definition of probability, we have

$$\phi\Delta = \frac{a}{m}, \quad \phi\Delta' = \frac{a'}{m}, \quad \&c.$$

But

$$\frac{a}{m} + \frac{a'}{m} + \frac{a''}{m} + \&c. = \frac{a + a' + a'' + \&c.}{m} = \frac{m}{m} = 1,$$

hence $\phi\Delta + \phi\Delta' + \phi\Delta'' + \&c. = 1.$

The probability that an error lies between certain limits is equal to the sum of all the values of $\phi\Delta$ between those limits. If the limits are infinitely near to each other, the value of $\phi\Delta$ may be considered constant; hence the probability that an error lies between Δ and $\Delta + d\Delta$, is $\phi\Delta\, d\Delta$.

It follows from this that the probability that an error lies between any limits, as a and b, is the sum of all the elements of the form $\phi\Delta\, d\Delta$ between those limits; that is,

$$\int_a^b \phi\Delta\, d\Delta.$$

But it must necessarily lie between $+\infty$ and $-\infty$, hence we have, Art. 72,

$$\int_{-\infty}^{+\infty} \phi\Delta\, d\Delta = 1 \qquad (115)$$

76. Let P = probability of the system of errors Δ, Δ', Δ'', &c., then, since the errors are independent of each other, we have by Art. 73,

$$P = \phi\Delta \cdot \phi\Delta' \cdot \phi\Delta'' \cdot \&c. \qquad (116)$$

To find the most probable system of errors, P must be made a maximum. Taking the logarithms of both sides of (116),

$$\log P = \log \phi\Delta + \log \phi\Delta' + \&c.$$

Differentiating and equating with 0,

$$\frac{d \log \phi\Delta}{d\Delta} \cdot \frac{d\Delta}{dx} + \frac{d \log \phi\Delta'}{d\Delta'} \cdot \frac{d\Delta'}{dx} + \&c. = 0.$$

But from (113),

$$d\Delta = -\,dx, \qquad d\Delta' = -\,dx, \qquad \&c.,$$

hence the last factor in each term is constant, and may be divided out. Denoting the other factors by $\phi'\Delta$, $\phi'\Delta'$, &c., the equation becomes

$$\phi'\Delta + \phi'\Delta' + \phi'\Delta'' + \&c. = 0,$$

which may be written in the form

$$\Delta \frac{\phi'\Delta}{\Delta} + \Delta' \frac{\phi'\Delta'}{\Delta'} + \Delta'' \frac{\phi'\Delta''}{\Delta''} + \&c. = 0,$$

which, compared with (114), shows that both cannot be true at the same time unless we have

$$\frac{\phi'\Delta}{\Delta} = \frac{\phi'\Delta'}{\Delta'} = \frac{\phi'\Delta''}{\Delta''} = \&c.,$$

that is, $\frac{\phi'\Delta}{\Delta} = \frac{d \log \phi\Delta}{\Delta\, d\Delta}$

is a constant. Denote it by K, then we have

$$d \log \phi\Delta = K\Delta\, d\Delta.$$

Integrating this, we find

$$\log \phi\Delta = K \frac{\Delta^2}{2} + \log C,$$

and passing from logarithms to numbers,

$$\phi\Delta = Ce^{\frac{1}{2} K\Delta^2},$$

in which e is the base of the Naperian system. In order that $\phi\Delta$ shall diminish as Δ increases, K must be negative. Putting $\frac{1}{2} K = -h^2$, we have

$$\phi\Delta = Ce^{-h^2\Delta^2} \tag{117}$$

hence by (115),

$$C \int_{-\infty}^{+\infty} e^{-h^2\Delta^2} d\Delta = 1.$$

But* $\int_{-\infty}^{+\infty} e^{-h^2\Delta^2} d\Delta = \frac{\sqrt{\pi}}{h}$, hence $C = \frac{h}{\sqrt{\pi}}$,

and (117) becomes

$$\phi\Delta = \frac{h}{\sqrt{\pi}} e^{-h^2\Delta^2} \tag{118}$$

the form of the function representing the probability of the error Δ.

* See the Author's "*Integral Examples*," No. 297.

THE MEASURE OF PRECISION.

77. The constant h in Eq. (118) measures the *precision* of the observations. For if the probability $(\phi\Delta)$ be constant, then if Δ is small, h will be large; that is, the better or more accurate the observations are, the larger will h be.

It follows from Art. 75, that in a given series of observations, the probability that the error of any one of them is between $\pm\,\delta$, is

$$\int_{-\delta}^{+\delta} \phi\Delta \, d\Delta,$$

or by (118),

$$\frac{h}{\sqrt{\pi}} \int_{-\delta}^{+\delta} e^{-h^2\Delta^2} \, d\Delta.$$

In another series, the probability that the error of an observation is between $\pm\,\delta'$ is

$$\frac{h'}{\sqrt{\pi}} \int_{-\delta'}^{+\delta'} e^{-h'^2\Delta^2} \, d\Delta$$

But since

$$\frac{h}{\sqrt{\pi}} \int_{-\delta}^{+\delta} e^{-h^2\Delta^2} \, d\Delta = \frac{1}{\sqrt{\pi}} \int_{-h\delta}^{+h\delta} e^{-h^2\Delta^2} \, d\,(h\Delta),$$

it appears that the above expressions are equal when $h\delta = h'\delta'$, If, for example, $h' = 2h$, the expressions become equal when $\delta = 2\delta'$, hence any error will have the same probability in the first system as half that error in the second, or, the precision of the second system is twice as great as that of the first.

THE CURVE OF PROBABILITY.

78. Since the probability of an error is a function of the error itself, we may regard (118) as the equation of a curve, taking Δ as the abscissa, and $\phi\Delta$ as the ordinate. The

value of h depends on the nature of the observations; making $h = 1$, (118) becomes

$$\phi\Delta = \frac{1}{\sqrt{\pi}} e^{-\Delta^2} \qquad (119)$$

from which the curve may be constructed. Its form is represented in Fig. 5.

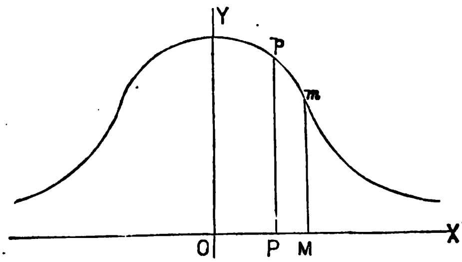

FIG. 5.

79. The position of the point where the curvature changes from concave to convex may be found by putting the second differential coefficient of the ordinate equal to 0.

Differentiating (119) twice, we find

$$\frac{d\,\phi\Delta}{d\Delta} = -\frac{2}{\sqrt{\pi}} \Delta e^{-\Delta^2};$$

$$\frac{d^2\,\phi\Delta}{d\,\Delta^2} = \frac{4\,\Delta^2}{\sqrt{\pi}} e^{-\Delta^2} - \frac{2}{\sqrt{\pi}} e^{-\Delta^2} = 0;$$

whence $$2\,\Delta^2 - 1 = 0,$$

and $$\Delta = OM = \sqrt{\tfrac{1}{2}} = 0.707 \qquad (120)$$

THE PRINCIPLE OF MINIMUM SQUARES.

80. If all the m observations of the series have equal precision, Eqs. (116) and (118) give

$$P = \phi\Delta \cdot \phi\Delta' \cdot \&c. = \frac{h^m}{\pi^{\frac{m}{2}}} e^{-h^2(\Delta^2 + \Delta'^2 + \Delta''^2 + \&c.)} \qquad (121)$$

and for the most probable system, P is a maximum; hence $\Delta^2 + \Delta'^2 + \Delta''^2 + \&c.$ is a minimum. Hence, the most

probable system of observations is that in which ***the sum of the squares of the errors is the least possible.***

But if all the observations are not of equal precision, h is not the same for all, and we have

$$P = \frac{h \cdot h' \cdot h'' \cdot \&c.}{\pi^{\frac{m}{2}}} e^{-(h^2 \Delta^2 + h'^2 \Delta'^2 + \&c.)},$$

and when P is a maximum, $h^2\Delta^2 + h'^2 \Delta'^2 + \&c.$ is a minimum; that is, in the most probable system, if each error is multiplied by its measure of precision, ***the sum of the squares of the products will be the least possible.***

The Probable Error.

81. The ***probable error*** is such a quantity that there is the same probability of the true error being greater, and of its being less than this.

If the errors of a series of observations are arranged in the order of their magnitude, without regard to their algebraic signs, and if r denote the error which stands exactly in the middle, then the number of errors less than r is equal to the number greater than r, hence the probability that the error of any observation is numerically less than r; that is, included between the limits $\pm r$, is $\frac{1}{2}$. The probability that it is greater than r is also $\frac{1}{2}$, hence r is the ***probable error.***

82. It has already been shown that the probability that an error lies between Δ and $\Delta + d\Delta$ is $\phi\Delta\, d\Delta$, or, by (118),

$$\frac{h}{\sqrt{\pi}} e^{-h^2 \Delta^2} d\Delta,$$

hence the probability that it lies between the limits 0 and t is

$$\frac{h}{\sqrt{\pi}} \int_0^t e^{-h^2 \Delta^2} d\Delta.$$

If we consider both positive and negative errors, the probability will be twice as great; denoting it by θ, we have

$$\theta = \frac{2\,h}{\sqrt{\pi}} \int_0^t e^{-h^2\Delta^2} d\Delta,$$

or putting $t = h\Delta$, whence

$$d\Delta = \frac{dt}{h},$$

$$\theta = \frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2} dt \qquad (122)$$

Developing e^{-t^2}, (122) becomes

$$\theta = \frac{2}{\sqrt{\pi}} \int_0^t dt \left(1 - t^2 + \frac{t^4}{1 \cdot 2} - \frac{t^6}{1 \cdot 2 \cdot 3} + \&c.\right)$$

and performing the integrations,

$$\theta = \frac{2}{\sqrt{\pi}} \left(t - \frac{t^3}{3} + \frac{1}{1 \cdot 2} \cdot \frac{t^5}{5} - \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{t^7}{7} + \&c.\right) \qquad (123)$$

which converges rapidly when t is small.

By assuming different values for t, and computing the values of θ, we may form the following table:

t	θ	t	θ
0.1	0.11246	0.6	0.60386
0.2	0.22270	0.7	0.67780
0.3	0.32863	0.8	0.74210
0.4	0.42839	0.9	0.79691
0.5	0.52050	1.0	0.84270

83. For the probable error, $\theta = \frac{1}{2} = 0.5$, and the corresponding value of t, denoted by T, found by interpolation from the table, is

$$T = 0.47694.$$

Since $h\Delta = t$, we have

$$hr = T = 0.47694,$$

or

$$r = \frac{0.47694}{h} \tag{124}$$

hence the probable error varies inversely as the constant h, and serves to measure the precision of the observations.

The probability that the error of an observation is less than the probable error r, is then, by (122),

$$\theta = \frac{2}{\sqrt{\pi}} \int_0^T e^{-t^2} dt = \tfrac{1}{2},$$

and the probability that it is less than nr is found by taking the same integral between the limits 0 and $nT = 0.47694\, n$. Thus, the probability that it is less than $\frac{1}{2}\, r$ is

$$\theta = \frac{2}{\sqrt{\pi}} \int_0^{0.23847} e^{-t^2} dt = 0.264;$$

hence in a series of 1000 observations, we should expect to find 264 errors less than half the probable error, and 500 less than the probable error. Continuing the calculation, we find there will probably be 688 errors less than $\frac{3}{2}\, r$, 823 less than $2\, r$, 957 less than $3\, r$, 993 less than $4\, r$, and 999 less than $5\, r$. It has been found in the case of large numbers of observations, that these results of theory agree very nearly with those of experience.

The Mean Error.

84. The *mean error* is the error whose square is the mean of the squares of all the errors.

Let it be denoted by ε, and put

$$\Delta^2 + \Delta'^2 + \Delta''^2 + \&c. = \Sigma\Delta^2,$$

then we have by the definition,

$$\varepsilon = \frac{\Sigma\Delta^2}{m} \tag{125}$$

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Eq. (121) may be written in the form

$$P = \frac{h^m}{\sqrt{\pi^m}} e^{-h^2 \Sigma\Delta^2},$$

in which P denotes the probability of a system of m observations, Δ, Δ', Δ'', &c., being the errors actually made. That value of h which renders P a maximum is the most probable value of h for the system. Putting the first differeniial coefficient of P equal to 0, we have

$$\frac{mh^{m-1}}{\pi^{\frac{m}{2}}} e^{-h^2 \Sigma\Delta^2} - \frac{2\, h^{m+1}}{\pi^{\frac{m}{2}}} \Sigma\Delta^2 \, e^{-h^2 \Sigma\Delta^2} = 0,$$

or

$$m - 2\, h^2 \, \Sigma\Delta^2 = 0,$$

whence

$$\frac{1}{2h^2} = \frac{\Sigma\Delta^2}{m} = \varepsilon^2 \qquad (126)$$

and

$$\varepsilon = \sqrt{\frac{\Sigma\Delta^2}{m}} = \frac{1}{h\sqrt{2}} \qquad (127)$$

and by (124),

$$\left.\begin{array}{l} \varepsilon = 1.4826\, r \\ r = 0.6745\, \varepsilon \end{array}\right\} \qquad (128)$$

or

When we make $h = 1$, (127) gives $\varepsilon = \sqrt{\tfrac{1}{2}} = 0.707$, whence we see by (120) that the mean error is represented by the abscissa OM, Fig. 5, and that the ordinate Mm represents its probability. The probable error is represented by the abscissa OP, and its probability by the ordinate Pp.

Practical Formulæ for Finding the Mean and Probable Errors.

85. Eq. (127) gives the correct mean error only when Δ, Δ', &c., are the actual errors of observation; that is, when the arithmetical mean of the observed values is the true value. Suppose it to differ from the true value by some quantity $\pm\,\delta$, then we must replace

$$\Delta \text{ by } \Delta \pm \delta,$$
$$\Delta^2 \text{ by } \Delta^2 \pm 2\,\Delta\delta + \delta^2,$$
$$\Sigma\Delta^2 \text{ by } \Sigma\Delta^2 \pm 2\,\Sigma\Delta \cdot \delta + m\delta^2,$$
$$\text{or by } \Sigma\Delta^2 + m\delta^2,$$

since, by (114), $\Sigma\Delta = 0$. Eq. (125) then gives

$$m\varepsilon^2 = \Sigma\Delta^2 + m\delta^2 \tag{129}$$

The value of the correction $m\delta^2$ cannot be accurately determined, but as the best approximation we may assume $m\delta^2 = \varepsilon^2$, hence $m\varepsilon^2 = \Sigma\Delta^2 + \varepsilon^2$, whence we find

$$\varepsilon^2 = \frac{\Sigma\Delta^2}{m-1}, \quad \text{or} \quad \varepsilon = \sqrt{\frac{\Sigma\Delta^2}{m-1}} \tag{130}$$

and by (128),

$$r = 0.6745\sqrt{\frac{\Sigma\Delta^2}{m-1}} \tag{131}$$

SOLUTION OF EQUATIONS OF CONDITION.

86. In the determination of astronomical and other data from observation, it usually happens that the quantities required are not themselves observed directly, but the quantity found by direct observation is a known function of those whose values are required. Each observation then gives rise to an equation between the quantity observed and those required. If the number of equations thus formed is just equal to that of the required quantities, their values can be found by elimination. Now, all the observations are liable to error, hence their number should be made as great as possible; but when the number of equations exceeds that of the unknown quantities, they cannot be solved without first combining them in such a manner as to make their number just equal to that of the unknown quantities. This may be done in many different ways, and it becomes an important question,—What particular combination of equations will give the most accurate results, that is, the smallest

probable errors? To answer this question is one principal object of the Method of Least Squares.

87. The equations derived from observation are of the form

$$ax + by + cz + \&c. + q = 0,$$

and are called *equations of condition*, because they express the conditions which the values of x, y, z, &c. are required to satisfy as nearly as possible. But since no values can be found which will satisfy them exactly, we should write instead of 0 in the second member, some quantity v, depending on the error of the observation which established the equation; our equations of condition are then of the form

$$\left.\begin{array}{l} ax + by + cz + \&c. + q = v \\ a'x + b'y + c'z + \&c. + q' = v' \\ a''x + b''y + c''z + \&c. + q'' = v'' \\ \qquad\qquad \&c. \end{array}\right\} \quad (132)$$

According to Article 80, the most probable values of x, y, z, &c., are those which substituted in (132) will render $v^2 + v'^2 + v''^2 + \&c.$, that is, Σv^2, a minimum. This being a function of the independent variables x, y, z, &c., the condition for a minimum requires that we have

$$\frac{d\Sigma v^2}{dx} = 0, \qquad \frac{d\Sigma v^2}{dy} = 0, \qquad \frac{d\Sigma v^2}{dz} = 0, \ \&c.,$$

that is,

$$\left.\begin{array}{l} v\dfrac{dv}{dx} + v'\dfrac{dv'}{dx} + v''\dfrac{dv''}{dx} + \&c. = 0 \\ v\dfrac{dv}{dy} + v'\dfrac{dv'}{dy} + v''\dfrac{dv''}{dy} + \&c. = 0 \\ v\dfrac{dv}{dz} + v'\dfrac{dv'}{dz} + v''\dfrac{dv''}{dz} + \&c. = 0 \\ \qquad \&c. \qquad\qquad \&c. \end{array}\right\} \quad (133)$$

But, differentiating Eqs. (132), we find

$$\frac{dv}{dx} = a, \qquad \frac{dv'}{dx} = a', \qquad \frac{dv''}{dx} = a'', \text{ \&c.,}$$

$$\frac{dv}{dy} = b, \qquad \frac{dv'}{dy} = b', \qquad \frac{dv''}{dy} = b'', \text{ \&c.,}$$

$$\text{\&c.} \qquad\qquad \text{\&c.}$$

Substituting these in (133), we have

$$av + a'v' + a''v'' + \text{\&c.} = 0,$$
$$bv + b'v' + b''v'' + \text{\&c.} = 0,$$
$$cv + c'v' + c''v'' + \text{\&c.} = 0, \quad \text{\&c.,}$$

or $\quad \Sigma (av) = 0, \quad \Sigma (bv) = 0, \quad \Sigma (cv) = 0, \quad$ &c.,

or substituting the value of v from (132),

$$\left.\begin{aligned} \Sigma (a^2)\, x + \Sigma (ab)\, y + \Sigma (ac)\, z + \text{\&c.} + \Sigma (aq) &= 0 \\ \Sigma (ab)\, x + \Sigma (b^2)\, y + \Sigma (bc)\, z + \text{\&c.} + \Sigma (bq) &= 0 \\ \Sigma (ac)\, x + \Sigma (bc)\, y + \Sigma (c^2)\, z + \text{\&c.} + \Sigma (cq) &= 0 \\ \text{\&c.} \qquad\qquad\qquad \text{\&c.} & \end{aligned}\right\} \quad (134)$$

the number of which is just equal to the number of unknown quantities.

Hence we have the following rule for combining equations of condition: *Multiply each equation of condition by the coefficient of x in that equation, and put the sum of the products equal to zero.* Doing the same with reference to each of the other unknown quantities, as many equations will thus be found as there are unknown quantities, from which their values can be found in the ordinary way.

The results obtained by applying this rule to the equations of condition are called *normal equations.*

88. The following example illustrates the application of the method, but in practice the number of equations should be very much increased if much dependence is to be placed on the results, since the theory of the method presupposes a large number of observations—large enough at least to determine the errors to which they are liable.

Given the following equations of condition:

$$
\begin{aligned}
x - y + 2z - 3 &= 0 \\
3x + 2y - 5z - 5 &= 0 \\
4x + y + 4z - 21 &= 0 \\
-x + 3y + 3z - 14 &= 0
\end{aligned}
$$

From these four equations we are to deduce three Normal Equations, the solution of which will give the most probable values of x, y and z.

Applying the rule, we multiply the first equation of condition by 1, the second by 3, the third by 4, and the fourth by -1; adding these products, we get for the first normal equation,

$$27x + 6y - 88 = 0,$$

and in a similar way we find the second to be

$$6x + 15y + z - 70 = 0,$$

and the third,

$$y + 54z - 107 = 0.$$

We now have three equations with three unknown quantities, the solution of which gives

$$z = 2.470, \qquad y = 3.551, \qquad z = 1.916.$$

CHAPTER VII.

THE TERRESTRIAL SPHEROID.

89. The earth being regarded as an oblate spheroid, let PEP', Fig. 6, be a section through its axis PP' and the point A, the place of the observer. Let P be the north pole, and E a point of the equator, and draw AO normal to the surface at A, then the line OA produced will meet the zenith of the observer, and the radius CA produced will meet the celestial sphere in a point called the ***geocentric zenith.***

Let $a = CE$ = equatorial radius,
$b = CP$ = polar radius,
$\rho = CA$ = radius at given place,
$\phi = AFB$ = latitude at given place,
$\phi' = ACB$ = geocentric latitude of do.,
$n = AF$ = normal at do.,
$s = FB$ = subnormal at do.,
$x = CB$, $y = AB$, the rectangular co-ordinates of the given place, the origin being at the earth's center.

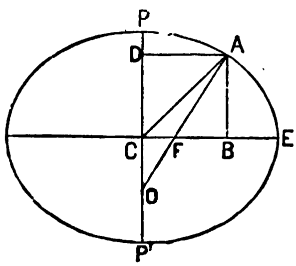

FIG. 6.

The latitude of the given place may be found by the methods explained in Chapter IV, hence in the problems which follow, ϕ will be regarded as known.

90. The meridian section PEP' being an ellipse, its equation is

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2) \tag{135}$$

or since
$$b^2 = a^2(1 - e^2) \tag{136}$$

$$y^2 = (1 - e^2)(a^2 - x^2) \tag{137}$$

in which e is the eccentricity. Differentiating this equation, and observing that y is a decreasing function of x, we have

$$2ydy = (1 - e^2)\,2xdx,$$

or
$$\frac{dy}{dx} = (1 - e^2)\frac{x}{y} \tag{138}$$

From the right triangle ABC, we have

$$\sin\phi' = \frac{y}{\rho} \quad (139), \qquad \cos\phi' = \frac{x}{o} \quad (140), \qquad \tan\phi' = \frac{y}{x} \quad (141)$$

and from the triangle ABF,

$$\cos\phi = \frac{s}{n} \quad (142), \qquad \tan\phi = \frac{y}{s} \quad (143)$$

91. *To find the geocentric latitude.*

The general formula for the subnormal is

$$s = y\frac{dy}{dx},$$

whence we have by (138),

$$s = x(1 - e^2) \tag{144}$$

hence (141) and (143) give

$$\frac{\tan\phi'}{\tan\phi} = \frac{s}{x} = 1 - e^2,$$

or
$$\tan\phi' = (1 - e^2)\tan\phi \tag{145}$$

The angle $CAO = AFE - ACF = \phi - \phi'$, and is called the Angle of the Vertical, or the Reduction of Latitude.

92. *To find the rectangular co-ordinates of the given place.*

Comparing Eqs. (141) and (145), we find

$$\frac{y}{x} = (1 - e^2) \tan \phi \tag{146}$$

whence $\qquad y^2 = x^2 (1 - e^2)^2 \tan^2 \phi,$

and by (137), $a^2 - x^2 = x^2 (1 - e^2) \tan^2 \phi,$

then
$$x^2 = \frac{a^2}{1 + (1 - e^2) \tan^2 \phi} \tag{147}$$

or
$$x^2 = \frac{a^2 \cos^2 \phi}{\cos^2 \phi + \sin^2 \phi - e^2 \sin^2 \phi},$$

and
$$x = \frac{a \cos \phi}{(1 - e^2 \sin^2 \phi)^{\frac{1}{2}}} \tag{148}$$

and by (146),
$$y = \frac{a\ (1 - e^2) \sin \phi}{(1 - e^2 \sin^2 \phi)^{\frac{1}{2}}} \tag{149}$$

To facilitate the application of logarithms, make
$$e \sin \phi = \sin \chi \tag{150}$$

then (148) and (149) become
$$x = a \sec \chi \cos \phi \tag{151}$$
$$y = a\ (1 - e^2) \sec \chi \sin \phi \tag{152}$$

93. *To find the radius of the earth at the given place.*

From (147) we have, by (145),
$$x^2 = \frac{a^2}{1 + \tan \phi \tan \phi'} = \frac{a^2 \cos \phi \cos \phi'}{\cos (\phi - \phi')} \tag{153}$$

and from (140),
$$\rho^2 = \frac{x^2}{\cos^2 \phi'} = \frac{a^2 \cos \phi}{\cos \phi' \cos (\phi - \phi')},$$

whence
$$\rho = a \sqrt{\frac{\cos \phi}{\cos \phi' \cos (\phi - \phi')}} \tag{154}$$

94. *To find the radius of curvature of the meridian.*

Let R = radius of curvature of the meridian at the point A. The expression for the radius of curvature of the ellipse is

8

$$R = \frac{(a^4y^4 + b^4x^4)^{\frac{3}{2}}}{a^4b^4} = \frac{\left(\frac{a^2}{b^2}\,y^2 + \frac{b^2}{a^2}\,x^2\right)^{\frac{3}{2}}}{ab}$$

and we have, by (135) and (136),

$$\frac{a^2}{b^2}\,y^2 = a^2 - x^2, \quad \frac{b^2}{a^2}\,x^2 = (1 - e^2)\,x^2, \quad ab = a^2\,(1 - e^2)^{\frac{1}{2}},$$

hence
$$R = \frac{(a^2 - e^2\,x^2)^{\frac{3}{2}}}{a^2\,(1 - e^2)^{\frac{1}{2}}}.$$

But from (148),

$$a^2 - e^2x^2 = a^2 - \frac{a^2e^2\cos^2\phi}{1 - e^2\sin^2\phi} = \frac{a^2 - a^2e^2}{1 - e^2\sin^2\phi},$$

whence we find

$$R = \frac{a\,(1 - e^2)}{(1 - e^2\sin^2\phi)^{\frac{3}{2}}} \tag{155}$$

which reduces, by (150), to

$$R = a\,(1 - e^2)\sec^3\chi \tag{156}$$

95. *To find the length of one degree of latitude.*

Let β = length of 1° of the meridian at the given place, then we have

$$\beta : 2\,\pi R :: 1^\circ : 360^\circ,$$

hence
$$\beta = \frac{\pi R}{180},$$

or by (155),

$$\beta = \frac{\pi R}{180} \cdot \frac{a\,(1 - e^2)}{(1 - e^2\sin^2\phi)^{\frac{3}{2}}} \tag{157}$$

or developing the denominator,

$$\beta = \frac{\pi}{180}\,a\,(1 - e^2)\,(1 + \tfrac{3}{2}\,e^2\sin^2\phi),$$

the 4th and higher powers of e being neglected.

If we put

$$\frac{\pi}{108} a (1 - e^2) = l \tag{158}$$

and
$$\tfrac{3}{2} e^2 l = q \tag{159}$$

we have
$$\beta = l + q \sin^2 \phi \tag{160}$$

At the equator, $\phi = 0°$, hence $\beta = l$; at the pole, $\phi = 90°$, and $\beta = l + q$.

96. *To find the compression of the earth.*

Let $c =$ the earth's compression, that is,

$$c = \frac{a - b}{a} = 1 - \frac{b}{a},$$

then
$$(1 - c)^2 = \frac{b^2}{a^2} = 1 - e^2 \tag{161}$$

whence
$$e^2 = 2\,c - c^2 = 2c, \quad \text{nearly.}$$

Hence
$$c = \tfrac{1}{2} e^2 \tag{162}$$

and by (159),
$$q = 3lc,$$

or
$$c = \tfrac{1}{3} \frac{q}{l} \tag{163}$$

97. *To find the radius and length of one degree of a parallel of latitude.*

Let $R' =$ radius,
and $\beta' =$ length of 1°, of the parallel of latitude ϕ. Since the radius of this parallel is $AD = x$, Fig. 6, we have by (150),

$$R' = a \sec \chi \cos \phi \tag{164}$$

whence
$$\beta = \frac{\pi a}{180} \cdot \sec \chi \cos \phi \tag{165}$$

98. The formulæ obtained in Articles 91 to 97 involve not only the latitude (ϕ) of the given place, but likewise (e), the eccentricity of the meridian, and (a), the equatorial radius. We are now prepared to show how the values of these constants may be found from the actual measurement of arcs of the meridian.

99. *To find by measurement the length of one degree of the meridian.*

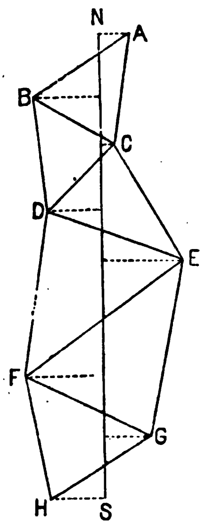

FIG. 7.

The following is the method of determining the exact length of an arc of the meridian: A base line AB, Fig. 7, is selected on a level plain, several miles in extent, and its length carefully measured by the most refined and accurate methods. A number of stations, C, D, E, &c., are also chosen as the vertices of a series of triangles extending in a north and south direction, these stations being so situated that in any one triangle the vertex of either angle can be seen from the other two. All the angles of each triangle are then carefully measured, and also the inclination of its sides to the true meridian. The angles being cleared from spherical excess, the sides of all the triangles are computed, beginning with ABC, of which the side AB was measured. Near the other extremity of the chain of triangles, another base, as GH, called a "verification base," is also measured, and unless there is a close agreement between its computed and measured length, the whole process is repeated.

The length of each side multiplied by the cosine of its inclination to the meridian, gives its projection on the meridian, and the sum of the projections of AB, BC, CD, &c., gives the length of the meridian line NS.

Let the latitudes of the extreme stations be found by observation;

let $\phi =$ half sum of the observed latitudes,

$d =$ their difference,

and $\beta =$ length of 1° of the meridian,

then $$d : 1^\circ :: NS : \beta,$$

whence $$\beta = \frac{NS}{d}.$$

Let these values of ϕ and β be substituted in Eq. (160). Two such measurements on different arcs will furnish two equations, from which the values of l and q can be found. It will be better, however, to increase the number of equations, and solve them by the Method of Least Squares.

The values of l and q being thus obtained, e becomes known from Eq. (159), and a from (158).

100. *Dimensions of the Earth.*

The values of β and ϕ derived from thirteen of the best measurements of an arc of the meridian in different latitudes, being substituted in (160), furnish thirteen equations of condition which, combined according to the rule in Art. 87, give two normal equations, from the solution of which we find

$$\left.\begin{aligned} l &= 68.7023 \text{ miles} \\ q &= 0.6878 \quad \text{"} \end{aligned}\right\} \qquad (166)$$

Hence the length of one degree of the meridian in miles, at the place whose latitude is ϕ, is, by (160),

$$\beta = 68.7023 + 0.6878 \sin^2 \phi \qquad (167)$$

At the equator, $\beta = 68.7023$ miles, and at the pole, $\beta = 69.39$ miles.

Equation (159) gives

$$e^2 = \frac{2q}{3l} = 0.00667437.$$

Equation (158) gives

$$a = \frac{180l}{\pi(1 - e^2)} = 3962.8025 \text{ miles.}$$

Equation (136) gives

$$b = a(1 - e^2)^{\frac{1}{2}} = 3949.5557 \text{ miles.}$$

Equation (162) gives

$$c = \tfrac{1}{2} e^2 = 0.003337 = \frac{1}{299.15}.$$

In the application of the formulæ obtained in Articles 91 to 97, the following logarithms will be convenient.

$$\log a = 3.5980024$$
$$\log b = 3.5965482$$
$$\log e = 8.9122052$$
$$\log (1 - e^2) = 9.9970916$$

CHAPTER VIII.

ECLIPSES OF THE SUN.*

101. Whenever at the time of new moon, the moon's shadow falls on the earth, at those places covered by the penumbra there will be seen a partial eclipse of the sun, while at those places covered by the umbra, or total shadow, there will be witnessed a total eclipse.

Now in the passage of the shadow eastward over the earth, an observer at any place in its path will be twice situated in the surface of the penumbral cone, and if also in the path of the umbra, he will be twice situated in the surface of the cone of total shadow. Whenever he is in the surface of the penumbra, as at C, Fig. 8, the discs of the sun and moon will appear in *exterior* contact; that is, he will see the beginning or end of the eclipse; and whenever he is in the surface of the umbra, as at C', Fig. 9, the discs will appear in *interior* contact; that is, he will see the beginning or end of the total phase. In the latter case, if the observer is beyond the vertex of the cone, as at C, Fig. 9, the eclipse will not be total, but annular, and the time of apparent interior contact will be the beginning or end of the annular phase.

* CHAUVENET'S *Spherical and Practical Astronomy*, Vol. I, Chap. X, contains an exhaustive analytical discussion of the whole subject of Eclipses, Occultations, Transits, etc., according to the elegant method of BESSEL. The materials from which the present Chapter was written were, for the most part, derived from that work.

Suppose a plane passed through the observer at right angles to the axis of the shadow, then it is evident that at the beginning and end of the eclipse, the observer's distance from the axis of the shadow is equal to the radius of the penumbra at that point; and also that at the times of beginning and end of the total or annular phase, the observer's distance from the axis is equal to the radius of the umbra at that point.

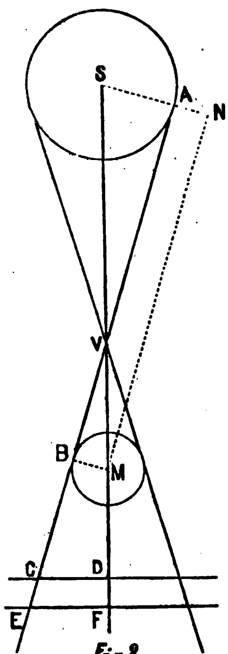

Fig. 8

In general, if we let

D = distance of observer from axis of shadow,

L = radius of shadow (umbra or penumbra),

then at the time of apparent contact of discs, we have

$$D = L \tag{168}$$

102. *To find the radius of the moon's shadow.*

Let the plane EF, Figs. 8 and 9, called the *principal plane*, be passed through the center of the earth perpendicular to the axis of the shadow, and therefore parallel to the plane CD through the observer.

Put $f = EVF =$ semi-angle of cone,

$p = VF$,

$w = DF$ = distance apart of planes,

$l = EF$ = radius of shadow on principal plane,

$L = CD$ = " " observer's plane,

From the triangle EVF, either figure, we have

$$\tan f = \frac{EF}{VF} = \frac{l}{p},$$

hence $$l = p \tan f \qquad (169)$$

and from the triangle CVD,

$$\tan f = \frac{CD}{VD} = \frac{L}{p - w},$$

whence $$L = (p - w) \tan f = l - w \tan f \qquad (170)$$

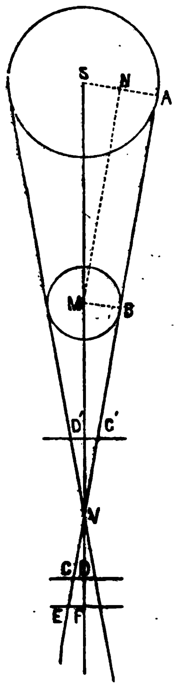

Fig. 9

Equations (169) and (170) give the radii of either the umbra or penumbra, according to which the angle f belongs.

In the case of total eclipse, the observer being at C', Fig. 2, $D'F = w$ is greater than $VF = p$, hence by (170), L will be negative; that is, *in a total eclipse, the radius of the umbra is to be considered negative.*

The radii of l and L are expressed in terms of p, f and w. Now we have, Figs. 8 and 9,

$$\tan f = \tan NMS = \frac{NS}{MS} = \frac{AS \pm BM}{MS};$$

$$\text{and } p = VF = MF \pm VM = MF \pm \frac{BM}{\sin f};$$

the upper signs being for the penumbra, and the lower for the umbra. Hence p and f are known when we know the radii of the sun and moon, and their distances from the earth's center. The value of $w = DF$ will be found hereafter.

103. *To find the position of the axis of the shadow.*

Let S and M, Fig 10, be the true places of the sun and moon, and S' and M' their apparent places as seen from the observer. Let them be referred to a

system of rectangular co-ordinates in space, having its origin at the center of the earth, the axis of Z being taken parallel to the axis of the shadow. The plane XY will then coincide with the principal plane, and the axis of X may be anywhere in this plane. Let it be the line joining the points in which the plane XY intersects the equator. The axis of Y will then be in the great circle containing the point Z, and the north pole; that is, in the hour circle of the point Z.

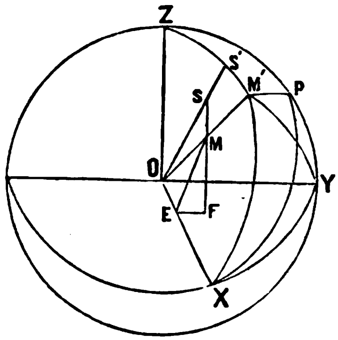

Fig. 10

Let x, y, z, be the co-ordinates of M, the moon's center, and $OM = r$, its distance from the origin, and let z be reckoned positive in the direction of MS, y positive towards the north, and x positive towards the east.

In the right-angled triangle MOE, we have

$$OE = OM \cos MOE,$$

$$\left.\begin{array}{lrl} \text{or} & x &= r \cos M'X \\ \text{and similarly,} & y &= r \cos M'Y \\ \text{and} & z &= r \cos MZ \end{array}\right\} \qquad (171)$$

Let α and δ denote the right ascension and declination of the moon, and a and d those of the point Z, then we have

R. A. of	$M' = \alpha$	Dec. of	$M' = \delta$
"	$Z = a$	"	$Z = d$
"	$Y = 180° + a$	"	$Y = 90° + d$
"	$X = 90° + a$	"	$X = 0°$

From this it follows that

$$\begin{aligned} MPZ &= \alpha - a, & PM' &= 90^\circ - \delta, \\ M'PY &= 180^\circ - (\alpha - a), & PZ &= 90^\circ - d, \\ MPX &= 90^\circ - (\alpha - a), & PY &= d, \\ & & PX &= 90^\circ. \end{aligned}$$

Now in the spherical triangle MPX,

$$\cos MX = \cos MP \cos PX + \sin MP \sin PX \cos MPX$$
$$= \cos \delta \sin (\alpha - a);$$

also, from the triangles MPY, MPZ,

$$\cos M'Y = \sin \delta \cos d - \cos \delta \sin d \cos (\alpha - a)$$
$$\cos MZ = \sin \delta \sin d + \cos \delta \cos d \cos (\alpha - a).$$

Substituting in (171) we obtain

$$\left.\begin{aligned} x &= r \cos \delta \sin (\alpha - a) \\ y &= r\,[\sin \delta \cos d - \cos \delta \sin d \cos (\alpha - a)] \\ z &= r\,[\sin \delta \sin d + \cos \delta \cos d \cos (\alpha - a)] \end{aligned}\right\} \quad (172)$$

Since the axis of Z is parallel to the axis of the shadow, the values of x and y for every point of the latter will be the same as for the moon's center.

104. *To find the distance of the place of observation from the axis of the shadow at a given time.*

Let the given place be referred to the same system of co-ordinates represented by Fig. 10. Let M now denote the position of the given place, then M' will be the geocentric zenith (Art. 89). Let u, v, w, be the co-ordinates of M referred to the axes of X, Y, Z, respectively, and let $\theta =$ given sidereal time.

The right ascension of the geocentric zenith, M', is equal to θ, (Art. 22); its declination equal to ϕ', the geocentric latitude; and the distance of the point M from the origin, equal to ρ, the radius of the place. Hence the values of u, v, w, are obtained from those of x, y, z, Eqs. 172, by replacing α by θ, δ by ϕ', and r by ρ, giving

$$\left.\begin{aligned} u &= \rho \cos \phi' \sin (\theta - a) \\ v &= \rho [\sin \phi' \cos d - \cos \phi' \sin d \cos (\theta - a)] \\ w &= \rho [\sin \phi' \sin d + \cos \phi' \cos d \cos (\theta - a)] \end{aligned}\right\} \quad (173)$$

The co-ordinates u, v, w, and the radius ρ, are expressed in terms of the equatorial radius, as unity.

Now since M, Fig. 10, is the geocentric zenith, the arc PM' is an arc of the meridian, hence the angle MPZ, or $(\theta - a)$, is the hour angle of the point Z at the given place. If we let μ = hour angle of point Z for meridian of Washington; and λ = longitude of given place from Washington (negative when east), we shall have

$$\theta - a = \mu - \lambda.$$

Substituting this in Eqs. (173), and putting

$$\rho \cos \phi' = h, \qquad \rho \sin \phi' = k,$$

we get

$$\left.\begin{aligned} u &= h \sin (\mu - \lambda) \\ v &= k \cos d - h \sin d \cos (\mu - \lambda) \\ w &= k \sin d + h \cos d \cos (\mu - \lambda) \end{aligned}\right\} \quad (174)$$

We find from (140) and (151), since we suppose $a = 1$,

$$\rho \cos \phi' = \sec \chi \cos \phi,$$

and from (139) and (152),

$$\rho \sin \phi' = (1 - e^2) \sec \chi \sin \phi,$$

hence we have

$$h = \sec \chi \cos \phi, \qquad k = (1 - e^2) \sec \chi \sin \phi \quad (175)$$

Now the co-ordinates of the place of observation, C, Figs. 8 and 9, being u, v, w, and those of the point D of the axis of the shadow, being x, y, z, in which $z = w = DF$, we have for the distance of the place of observation from the axis of the shadow,

$$D = \sqrt{(x - u)^2 + (y - v)^2}.$$

Substituting this value of D, and that of L from Eq. (170), in (168), and squaring both members, we obtain

$$(x - u)^2 + (y - v)^2 = (l - w \tan f)^2 \qquad (176)$$

the fundamental equation of eclipses.

105. *To adapt the fundamental equation to the Eclipse Tables of the American Ephemeris.*

The Tables of Data for Solar Eclipses, given in each volume of the American Ephemeris, are based on the following transformation of BESSEL'S fundamental equation, suggested by Prof. T. H. SAFFORD. The equation (176) may be written in the form

$$(x - u)^2 = (l - w \tan f)^2 - (y - v)^2$$
$$= [(l - w \tan f) + (y - v)]\,[(l - w \tan f) - (y - v)].$$

Putting

$$\left.\begin{aligned} a &= x - u \\ b &= (l - w \tan f) + (y - v) \\ c &= (l - w \tan f) - (y - v) \end{aligned}\right\} \qquad (177)$$

we have

$$a^2 = bc \qquad (178)$$

in which, from (174),

$$\left.\begin{aligned} a &= x - h \sin (\mu - \lambda) \\ b &= l + y - k (\cos d + \sin d \tan f) \\ &\quad + h (\sin d - \cos d \tan f) \cos (\mu - \lambda) \\ c &= l - y + k (\cos d - \sin d \tan f) \\ &\quad - h (\sin d + \cos d \tan f) \cos (\mu - \lambda) \end{aligned}\right\} \qquad (179)$$

These formulæ are adapted to the Tables of Data in the Ephemeris by putting

$$A = x, \qquad B = l + y, \qquad C = -l + y,$$
$$E = \cos d + \sin d \tan f,$$
$$F = \cos d - \sin d \tan f,$$
$$G = \sin d - \cos d \tan f.$$
$$H = \sin d + \cos d \tan f,$$

all of which are independent of the place of observation, and, together with the value of μ, are given for every 10 minutes of Washington time, for each eclipse.

Eqs. (179) now become

$$\left.\begin{aligned} a &= A - h \sin(\mu - \lambda) \\ b &= B - Ek + Gh \cos(\mu - \lambda) \\ c &= -C + Fk - Hh \cos(\mu - \lambda) \end{aligned}\right\} \quad (180)$$

which agree with the formulæ in the Nautical Almanac.

106. *To compute the time of occurrence of either phase of a solar eclipse.*

The problem can only be solved by successive approximations. Take the time of the given phase as nearly as possible from the Eclipse Chart, and for this time take the values of μ, A, B, C, and the logarithms of E, F, G, H, from the tables. Compute h and k by (175), and a, b, c, by (180). If the assumed time was correct, Eq. (178) will be satisfied, and we shall find

$$a = \sqrt{bc}.$$

If we do not, we proceed to find the correction for the assumed time. Put

$$\sqrt{bc} = m;$$

and let a', b', m', denote the changes of a, b, m, in one second, and t the required correction; then at the true time of the phase we shall have

$$a + ta' = m + tm',$$

whence

$$t = \frac{m - a}{a' - m'} \quad (181)$$

To find a', differentiate the first of Eqs. (180), in which h and λ are constants, then

$$da = dA - h \cos(\mu - \lambda)\, d\mu,$$

that is,

$$a' = A' - \mu' h \cos(\mu - \lambda) \quad (182)$$

in which a', A', μ', are the changes of a, A, μ, in one second.

We also find by differentiation, since the changes of l, f, E, F, G, H, are so small that they may be regarded as constant,

$$A' = x', \qquad B' = y' = C',$$

$$b' = B' - \mu' Gh \sin(\mu - \lambda) \tag{183}$$

$$c' = -C + \mu' Hh \sin(\mu - \lambda) \tag{184}$$

But as f is a very small angle, we may put $G = H$, then $b' = -c'$, and since $m^2 = bc$, we have

$$2mdm = c \cdot db + b \cdot dc,$$

or

$$2mm' = cb' + bc' = (c - b)\, b' \tag{185}$$

whence

$$m' = \tfrac{1}{2}\left(\frac{c}{m} - \frac{b}{m}\right) b' = \tfrac{1}{2}\left(\sqrt{\frac{c}{b}} - \sqrt{\frac{b}{c}}\right) b'.$$

Let us assume

$$\tan \tfrac{1}{2} Q = \sqrt{\frac{c}{b}} = \frac{c}{m} = \frac{m}{b} \tag{186}$$

then $\quad m' = \tfrac{1}{2}(\tan \tfrac{1}{2} Q - \cot \tfrac{1}{2} Q)\, b' = -b' \cot Q \quad$ (187)

and by (181),

$$t = \frac{m - a}{a + b' \cot Q} \tag{188}$$

The values of A', B' and C' are very small, and are given in the tables in terms of 0.000001 as the unit, so that the value of t, Eq. (188), requires to be multiplied by $10^6 = 1000000$. The value of μ' is constant, and its logarithm is 1.86167.

These formulæ apply equally well to the computation of the beginning or end of the eclipse, and of the beginning or end of the total or annular phase, but in the former case the data for the penumbra must be used, and in the latter the data for the shadow.

107. *To find the points of first and last contact.*

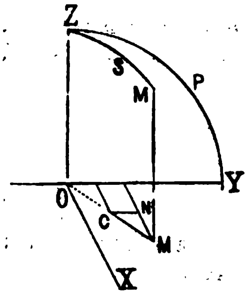

Fig. 11.

Suppose OX, OY and OZ, Fig. 11, to represent the same system of co-ordinate axes as in Fig. 10, and let M be the projection of the moon's center, and C that of the place of observation, on the plane XY; then MM' represents the axis of the moon's shadow, and $CM = D$. Drawing MN and CN parallel to OX and OY, respectively, we have,

$$MN = x - u, \qquad CN = y - v.$$

Denote the angle $MCN = M'ZP$, by ψ, then we have

$$D \sin \psi = x - u \tag{189}$$

$$D \cos \psi = y - v \tag{190}$$

whence

$$\tan \psi = \frac{x - u}{y - v} \tag{191}$$

Equation (187) gives, by (185),

$$\tan Q = -\frac{b'}{m'} = \frac{2m}{b - c} = \frac{2a}{b - c},$$

since at the instant of contact, $m = a$. Hence by (177),

$$\tan Q = \frac{x - u}{y - v},$$

and by (191),

$$Q = \psi \tag{192}$$

Now the arc ZSM' passing through the centers of the sun and moon, must pass through their point of contact, hence the angle ψ, or PZM', is very nearly the same as the angle between the sun's hour circle and the line joining its center and the point of contact; that is, the angular distance of the point of contact from the north point, of the sun's disc, reckoned towards the east. Hence by (192), the value of Q obtained from (186) in computing the time of

beginning may be taken as the angle of first contact from the north point, and the value of Q obtained in computing the time of end, as the angle of last contact from the north point.

We see from (189) that if D is *positive*, the sign of ψ will be the same as that of $x - u$; that is, *Q will have the same sign as a.* Now we have seen that in the case of a total eclipse, the radius, L, is negative, hence D, in Eq. (189), must be regarded as negative for a total eclipse, that is, ψ will have a sign contrary to that of $x - u$. Hence in computing the beginning and end of the *total phase*, Q must be taken with a *sign contrary to that of a.*

108. The points of first and last contact are generally determined by their angular distances from the sun's vertex, that is, the point nearest the zenith.

Let $V =$ angular distance from vertex, and $S =$ parallactic angle, PSZ, Fig, 2; then we have

$$V = Q - S \tag{193}$$

To find S, take the general equations of Spherical Trigonometry,

$$\sin a \sin C = \sin c \sin A,$$

$$\sin a \cos C = \cos c \sin b - \sin c \cos b \cos A,$$

and apply them to the triangle PZS, Fig. 2, making

$A = P,\ \ C = S,\ \ a = 90° - h,\ \ b = 90° - \delta,\ \ C = 90° - \phi,$

and calling the sun's hour angle $\mu - \lambda$, as it is very nearly; we thus find

$$\cos h \sin S = \cos \phi \sin (\mu - \lambda),$$

$$\cos h \cos S = \sin \phi \cos \delta - \cos \phi \sin \delta \cos (\mu - \lambda).$$

Putting $\qquad \sin \phi = p \sin P,$

and $\qquad \cos \phi \cos (\mu - \lambda) = p \cos P,$

we have $\begin{cases} \cos h \sin S = p \cos P \tan (\mu - \lambda) \\ \cos h \cos S = p \sin (P - \delta), \end{cases}$

whence we find

$$\tan P = \frac{\tan \phi}{\cos (\mu - \lambda)} \tag{194}$$

and

$$\tan \mathcal{S} = \frac{\cos P \tan (\mu - \lambda)}{\sin (P - \delta)} \tag{195}$$

The angle V, found by (193), is for direct vision; if an inverting telescope be used, it must be increased or decreased by 180°.

109. *To determine the Longitude of the place from the observation of a solar eclipse.*

The observation gives us the local sidereal or mean solar time of the observed contact.

Let s = local sidereal time of observed contact,
t = local mean time of do.,
T_1 = Washington mean time of do.,
T_2 = assumed Washington time for which the odserved phase was computed,
s_2 = corresponding Washington sid. time,
$\tau = T_1 - T_2$, expressed in hours,
λ = west longitude of the place.

Then $T_1 = t + \lambda$, and $\tau = t + \lambda - T_2$,

whence

$$\lambda = T_2 - t + \tau \tag{196}$$

or if τ is in *sidereal* time,

$$\lambda = s_2 - s + \tau \tag{197}$$

In order to find T_1, the longitude of the place must be approximately known.

110. It remains to find an expression for τ in terms of the data furnished by the Ephemeris. Since we have put, Art. 105,

$$x = A, \qquad l + y = B, \qquad -l + y = C,$$

whence

$$y = \tfrac{1}{2}(B + C),$$

we have from (177),

$$u = A - a; \qquad b - c = 2(y - v),$$

whence $v = y - \frac{1}{2}(b - c) = \frac{1}{2}(B + C) - \frac{1}{2}(b - c);$

$$b + c = 2(l - w \tan f) = 2L,$$

whence $L = \frac{1}{2}(b + c).$

Hence from the previous computation for the time T_2, we have

$$x_2 = A_2, \qquad y_2 = \frac{1}{2}(B_2 + C_2) \tag{198}$$

and after computing a_1, b_1, c_1, for the time T_1, by Eqs. (180), we shall have

$$\left.\begin{aligned} u_1 &= A_1 - a_1 \\ v_1 &= \tfrac{1}{2}(B_1 + C_1) - \tfrac{1}{2}(b_1 - c_1) \\ L_1 &= \tfrac{1}{2}(b_1 + c_1) \end{aligned}\right\} \tag{199}$$

Let x' and y' denote the hourly changes of x and y at the time T_1, then

$$x_1 = x_2 + \tau x', \qquad y_1 = y_2 + \tau y',$$

and since at the time of contact, T_1, Eq. (168) is satisfied, (189) and (190) become

$$L_1 \sin\psi = x_1 - u_1, \qquad L_1 \cos\psi = y_1 - v_1,$$

or

$$\left.\begin{aligned} L_1 \sin\psi &= x_2 - u_1 + \tau x' \\ L_1 \cos\psi &= y_2 - v_1 + \tau y' \end{aligned}\right\} \tag{200}$$

in which τ is the only unknown quantity. Put.

$$x_2 - u_1 = m \sin M, \qquad x' = n \sin N,$$

$$y_2 - v_1 = m \cos M, \qquad y' = n \cos N,$$

whence $\tan M = \dfrac{x_2 - u_1}{y_2 - v_1}, \qquad \tan N = \dfrac{x'}{y'}$ (201)

$$m = \frac{x_2 - u_1}{\sin M} = \frac{y_2 - v_1}{\cos M}, \qquad n = \frac{x'}{\sin N} = \frac{y'}{\cos N} \tag{202}$$

then (200) become

$$L_1 \sin\psi = m \sin M + \tau n \sin N,$$

$$L_1 \cos\psi = m \cos M + \tau n \cos N.$$

Multiply the first by $\cos N$, and the second by $\sin N$, and subtract, then

$$L_1 \sin (\psi - N) = m \sin (M - N) \tag{203}$$

Multiply the first by $\sin N$, and the second by $\cos N$, and add, then

$$L_1 \cos (\psi - N) = m \cos (M - N) + n\tau \tag{204}$$

If we put $\psi - N = \varepsilon$, (203) gives

$$\sin \varepsilon = \frac{m \sin (M - N)}{L_1} \tag{205}$$

and (204) gives $\tau = \dfrac{L_1 \cos \varepsilon}{n} - \dfrac{m \cos (M - N)}{n}$,

but from (205), $L_1 = \dfrac{m \sin (M - N)}{\sin \varepsilon}$,

whence

$$\tau = \frac{m}{n} \cdot \frac{\sin (M - N) \cos \varepsilon}{\sin \varepsilon} - \frac{m}{n} \cdot \frac{\cos (M - N) \sin \varepsilon}{\sin \varepsilon}$$

$$= \frac{m}{n} \cdot \frac{\sin (M - N - \varepsilon)}{\sin \varepsilon} \tag{206}$$

Eq. (205) gives two values of ε, and that value must be taken in (206) which will render

$$\frac{L_1 \cos \varepsilon}{n}$$

negative for first contact, and positive for last contact, recollecting that L_1 is negative in the case of a total eclipse.

Equation (206) gives τ in mean solar hours; it may be reduced to seconds, if the longitude is found by (196), by multiplying by 3600, but if by (197), by multiplying by 3609.856.

This method of finding the longitude involves the theoretical inaccuracy of using the approximate longitude in finding T_1, on which the values of u_1, v_1, x' and y' depend, but it will generally be known with sufficient accuracy for this purpose. Any errors in the data derived from the Ephemeris will of course affect the resulting longitude.

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